

## EFFECTIVE HEIGHT UPPER BOUNDS ON ALGEBRAIC TORI

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The main emphasis will be on height upper bounds in the algebraic torus  $\mathbf{G}_m^n$ . By height we will mean the absolute logarithmic Weil height. Section 3.2 contains a precise definition of this and other more general height functions. The first appendix gives a short overview of known results in the abelian case. The second appendix contains a few height bounds in Shimura varieties.

## 1. A BRIEF HISTORICAL OVERVIEW IN THE TORIC SETTING

In 1999, Bombieri, Masser, and Zannier proved the following result.

**Theorem 1** (Bombieri, Masser, and Zannier [6]). *Let  $C$  be an irreducible algebraic curve inside  $\mathbf{G}_m^n$  and defined over  $\overline{\mathbf{Q}}$ , an algebraic closure of  $\mathbf{Q}$ . Suppose that  $C$  is not contained in a proper coset, that is the translate of a proper algebraic subgroup. Then the height of points on  $C$  that are contained in a proper algebraic subgroup is bounded from above uniformly.*

The adjective uniformly refers to the fact that the bound for the height does not depend on the algebraic subgroup. It may and will depend on the curve  $C$ .

Using this height upper bound together with height lower bounds in the context of Lehmer’s question they were able to prove the following theorem.

**Theorem 2** (Bombieri, Masser, and Zannier [6]). *Let  $C$  be as in Theorem 1. There are only finitely many points on  $C$  that are contained in an algebraic subgroup of codimension at least 2.*

Heuristically, points considered in this theorem are rare since they are inside the *unlikely intersection* of a curve and a subvariety of codimension 2. The difficulty in proving such a theorem arises from the fact that the codimension 2 subvarieties varies over an infinite family.

**Example.** We exemplify this on the example of a line in  $\mathbf{G}_m^3$ . The line is parametrized by

$$(1.1) \quad (\alpha t + \alpha', \beta t + \beta', \gamma t + \gamma')$$

with  $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \overline{\mathbf{Q}}$  fixed. To guarantee that our line is not contained in a proper coset of  $\mathbf{G}_m^3$  we ask that

$$\alpha\beta\gamma \neq 0 \quad \text{and} \quad \frac{\alpha'}{\alpha}, \frac{\beta'}{\beta}, \frac{\gamma'}{\gamma} \text{ are pair-wise distinct.}$$

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Theorem 1 implies that there is a constant  $B$  with the following property. Suppose  $t \in \overline{\mathbf{Q}}$  such that no coordinate of (1.1) vanishes and such that there exist  $(a, b, c) \in \mathbf{Z}^3$  with

$$(1.2) \quad (\alpha t + \alpha')^a (\beta t + \beta')^b (\gamma t + \gamma')^c = 1$$

then the height of  $t$  is at most  $B$ .

The exponent vector  $(a, b, c)$  is allowed to vary in (1.2). It determines a proper algebraic subgroup of  $\mathbf{G}_m^3$ . Conversely, any such subgroup arises in this way.

In order to obtain finiteness of the set of  $t$  as in the second result of Bombieri, Masser, and Zannier we must impose a second condition. More precisely, there are only finitely many  $t$  with (1.2) such that there is  $(a', b', c') \in \mathbf{Z}^3$  linearly independent of  $(a, b, c)$  with

$$(\alpha t + \alpha')^{a'} (\beta t + \beta')^{b'} (\gamma t + \gamma')^{c'} = 1.$$

The two monomial equations determined by  $(a, b, c)$  and  $(a', b', c')$  define an algebraic subgroup of  $\mathbf{G}_m^3$  of codimension 2.

Bombieri, Masser, and Zannier remarked that any curve contained in a proper coset invariable leads to unbounded height. So this condition cannot be dropped from their theorem. On the other hand, it remained unclear if the restriction was necessary in order to achieve finiteness in the situation of unlikely intersections. The authors posed the following question.

*Question 1.* Suppose  $C \subset \mathbf{G}_m^n$  is an irreducible algebraic curve defined over  $\overline{\mathbf{Q}}$  that is not contained in a proper algebraic subgroup. Is the set of points on  $C$  that are contained in an algebraic subgroup of codimension 2 finite?

First progress was made in 2006 when the same group of authors obtained a partial answer. By making a detour to surfaces, Bombieri, Masser, and Zannier gave a positive answer to their question in low dimension.

**Theorem 3** (Bombieri, Masser, and Zannier [7]). *Suppose  $n \leq 5$ . Let  $C$  be an irreducible algebraic curve inside  $\mathbf{G}_m^n$  and defined over  $\overline{\mathbf{Q}}$ . Suppose that  $C$  is not contained in a proper algebraic subgroup. There are only finitely many points on  $C$  that are contained in an algebraic subgroup of codimension at least 2.*

In the most interesting case  $n = 5$  they constructed an algebraic surface  $S \subset \mathbf{G}_m^3$  derived from the curve. In order to answer their question for  $C$ , a bounded height result akin to Theorem 1 was needed for points on  $S$  lying in an algebraic subgroup of codimension  $\dim S = 2$ . So the algebraic subgroups in question have dimension at most 1. Luckily, height bounds on the intersection a fixed variety with varying algebraic subgroups of dimension 1 can be handled by early work of Bombieri and Zannier on subvarieties  $X \subset \mathbf{G}_m^n$  of unrestricted dimension.

**Theorem 4** (Bombieri and Zannier [47]). *Suppose  $X \subset \mathbf{G}_m^n$  is an irreducible algebraic subvariety defined over  $\overline{\mathbf{Q}}$ . Let  $X^\circ$  be the complement in  $X$  of the union of all cosets of positive dimension that are contained in  $X$ . Then the height of points on  $X$  that are contained in an algebraic subgroup of dimension at most 1 is uniformly bounded.*

The construction used by the three authors works for  $n > 5$  too and always yields a surface  $S$ . Unfortunately, it will not be a hypersurface in general. Thus the algebraic subgroups, having codimension  $\dim S$ , are too large for Theorem 4.

One interesting aspect of Theorem 3 is that its proof is effective, provided an effective height bound in Theorem 4 is available. The author [20] made a generalization of Theorem 4 effective and completely explicit. He gave a height bound for the points considered in Theorem 4 in terms of  $n$  and the degree and height of  $X$ .

In 2008, and using a different approach, Maurin gave a positive answer to the question posed above for all  $n$ .

**Theorem 5** (Maurin [27]). *Let  $C$  be an irreducible algebraic curve inside  $\mathbf{G}_m^n$  and defined over  $\overline{\mathbf{Q}}$ . Suppose that  $C$  is not contained in a proper algebraic subgroup. There are only finitely many points on  $C$  that are contained in an algebraic subgroup of codimension at least 2.*

Maurin first proves that the height is bounded from above. But his method differed substantially from the one used in the proof of Theorem 1. It relied on generalization of Vojta's inequality by Rémond [35] which is able to account for the varying algebraic subgroups. The original Vojta inequality appeared prominently in a proof of Faltings' Theorem, the Mordell Conjecture.

Maurin's Theorem holds for curves defined over  $\overline{\mathbf{Q}}$ . But its statement makes no reference to algebraic numbers. Having it at their disposal Bombieri, Masser, and Zannier applied specialization techniques to obtain the following result.

**Theorem 6** (Bombieri, Masser, and Zannier [10]). *Let  $C$  be an irreducible algebraic curve inside  $\mathbf{G}_m^n$  and defined over  $\mathbf{C}$ . Suppose that  $C$  is not contained in a proper algebraic subgroup. There are only finitely many points on  $C$  that are contained in an algebraic subgroup of codimension at least 2.*

For curves this theorem implies Zilber's Conjecture 1 [49] on anomalous intersections, also dubbed the *Conjecture on Intersection with Tori* or short CIT. It also resolves, in the case of curves, the related Torsion Finiteness Conjecture stated by Bombieri, Masser, and Zannier [8].

In the mean time, progress was being made on boundedness of height for surfaces by Bombieri, Masser, and Zannier. A statement of their result requires the definition of the *open anomalous* set  $X^{\text{oa}}$  of an irreducible closed subvariety  $X \subset \mathbf{G}_m^n$ . We will give a proper definition further down near (2.1).

If  $C$  is a curve then  $C^{\text{oa}} = C$  if and only if  $C$  is not contained in a proper coset. Otherwise, we will have  $C^{\text{oa}} = \emptyset$ . So Theorem 1 can be reformulated succinctly follows. Say  $C \subset \mathbf{G}_m^n$  is an irreducible algebraic curve defined over  $\mathbf{Q}$ . Then any point on  $C^{\text{oa}}$  contained in a proper algebraic subgroup has height bounded from above uniformly.

**Theorem 7** (Bombieri, Masser, and Zannier [9]). *Let  $P$  be a plane inside  $\mathbf{G}_m^n$  and defined over  $\overline{\mathbf{Q}}$ . The open anomalous set  $P^{\text{oa}}$  is Zariski open in  $P$ . The height of points on  $P^{\text{oa}}$  that are contained in an algebraic subgroup of codimension at least 2 is bounded from above uniformly.*

Unfortunately, for  $X$  of dimension strictly greater than 1 the anomalous  $X^{\text{oa}}$  cannot be described as easily as for curves. Their complete result also contains a precise description of  $P^{\text{oa}}$  and in particular a necessary and sufficient condition for  $P^{\text{oa}} \neq \emptyset$ .

These three authors expected boundedness of height to hold for general subvarieties of  $\mathbf{G}_m^n$  and thus formulated the Bounded Height Conjecture [8]. Heuristic observations

dictate the algebraic subgroups of have dimension complementary to the dimension of the fixed variety. In other words, if the subvariety of  $\mathbf{G}_m^n$  has dimension  $r$ , one must intersect with algebraic subgroups of codimension  $r$ .

For arbitrary  $X$ , the following result of Bombieri, Masser, and Zannier controls  $X^{\text{oa}}$ .

**Theorem 8** (Bombieri, Masser, and Zannier [8]). *Let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\mathbf{C}$ . Then  $X^{\text{oa}}$  is Zariski open in  $X$ .*

In fact they prove a stronger structure theorem for  $X^{\text{oa}}$ .

Later in 2009, the author showed boundedness of height for arbitrary subvarieties of  $\mathbf{G}_m^n$ .

**Theorem 9** ([22]). *Let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$ . The height of points on  $X^{\text{oa}}$  that are contained in an algebraic subgroup of codimension at least  $\dim X$  is bounded from above uniformly.*

This height bound is now sufficiently general to prove height bounds on surfaces arising from the strategy devised by Bombieri, Masser, and Zannier in their proof of Theorem 3. In joint work with this group, the author gave a new proof of Maurin's Theorem [5].

The purpose of this article is to provide an effective and fully explicit version of Theorem 9. In combination with the methods presented in [5] we obtain an effective version of Maurin's Theorem.

In recent work, Maurin obtained a height bound when the algebraic subgroups in question have codimension at least  $1 + \dim X$  but are enlarged by a subgroup of finite type.

**Theorem 10** (Maurin [28]). *Let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  and let  $\Gamma \subset \mathbf{G}_m^n(\overline{\mathbf{Q}})$  be the division closure of a finitely generated subgroup. The set of points  $ab \in X^{\text{oa}}$  with  $a \in \Gamma$  and  $b$  in an algebraic subgroup of codimension at least  $1 + \dim X$  has uniformly bounded height.*

From this height bound, he is able to deduce the non-Zariski denseness of the points  $ab$  in question.

## 2. AN EFFECTIVE HEIGHT BOUND

In order to state our result we must define the anomalous locus of an irreducible closed subvariety  $X \subset \mathbf{G}_m^n$  defined over  $\mathbf{C}$ .

Let  $s \geq 0$  be an integer. An irreducible closed subvariety  $Z \subset X$  is called  $s$ -anomalous if there exists a coset  $H \subset \mathbf{G}_m^n$  containing  $Z$  such that

$$\dim Z \geq \max\{1, s + \dim H - n + 1\}.$$

If  $s = \dim X$  then  $Z$  is called anomalous. The  $(s)$ -anomalous locus of  $X$  is the union of all  $(s)$ -anomalous subvarieties of  $X$ . We define  $X^{\text{oa},[s]}$  to be the complement of the  $s$ -anomalous locus and set

$$(2.1) \quad X^{\text{oa}} = X^{\text{oa},[\dim X]}.$$

We also set

$$(\mathbf{G}_m^n)^{[s]} = \bigcup_{\substack{H \subset \mathbf{G}_m^n \\ \text{codim } H \geq s}} H(\overline{\mathbf{Q}})$$

where the union is over all algebraic subgroups  $H$  of  $\mathbf{G}_m^n$  that have codimension at least  $s$ . We remark that all algebraic subgroups of  $\mathbf{G}_m^n$  are defined over  $\mathbf{Q}$ .

Our main result is the following explicit version of the Bounded Height Theorem.

**Theorem 11.** *Let  $n \geq 1$  and let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $r$  and say  $s \geq 0$  is an integer. We set  $C = (600n^7(2n)^{n^2})^r$ . There exists a Zariski closed subset  $Z \subset X$  with the following properties.*

- (i) *We have  $X^{\text{oa},[s]} \subset X \setminus Z$ .*
- (ii) *Each irreducible component  $V$  of  $Z$  satisfies*

$$\deg(V) \leq (2\deg(X))^C$$

*and*

$$h(V) \leq (2\deg(X))^C(1 + h(X)).$$

*The number of irreducible components of  $Z$  is at most  $(2\deg(X))^C$ .*

- (iii) *If  $p \in (X \setminus Z)(\overline{\mathbf{Q}}) \cap (\mathbf{G}_m^n)^{[s]}$ , then*

$$h(p) \leq (2\deg(X))^C(1 + h(X)).$$

The general proof strategy is similar to the one used by the author to prove the original Bounded Height Conjecture. For example, Ax's Theorem [1] still plays a pivotal role. Although there is one significant different. The first proof relied ultimately on a compactness argument to bound from below a certain intersection number. Therefore, no effective or even explicit result could be attained. In the current proof we use instead a recent result of Sturmfels and Tevelev [44] from Tropical Geometry to make everything explicit.

Going the other way, we obtain an amusing corollary on the tropicalization  $\mathcal{T}(X) \subset \mathbf{Q}^n$  of an irreducible closed subvariety of  $\mathbf{G}_m^n$  defined over  $\mathbf{C}$ . The proof and necessary definitions are given in Section 8. It suffices to say at this point that  $\mathcal{T}(X)$  is a finite union of rational polyhedral cones of dimension  $\dim X$ . We let  $\overline{\mathcal{T}(X)}$  denote the topological closure of  $\mathcal{T}(X)$  inside  $\mathbf{R}^n$ . The following corollary shows that the configuration of cones determining  $\overline{\mathcal{T}(X)}$  satisfy a rationality condition with respect to projecting to  $\mathbf{R}^{\dim X}$ . It will not play a role in the proof of our main result.

**Corollary 1.** *Let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\mathbf{C}$  of dimension  $r$ . If there is  $\varphi_0 \in \text{Mat}_{r,n}(\mathbf{R})$  of rank  $r$  with  $\varphi_0(\overline{\mathcal{T}(X)}) \neq \mathbf{R}^r$  then there exists  $\varphi \in \text{Mat}_{r,n}(\mathbf{Q})$  of rank  $r$  with  $\varphi(\mathcal{T}(X)) \neq \mathbf{Q}^r$ .*

This rationality property is non-trivial as portrayed by an example in Section 8. There we exhibit four two-dimensional vector subspaces  $L_{1,2,3,4} \subset \mathbf{Q}^4$  and a surjective  $\varphi_0 \in \text{Mat}_{2,4}(\mathbf{R})$  such that  $\dim \varphi_0(\overline{L_i}) < 2$  for  $1 \leq i \leq 4$  but with

$$\varphi(L_1 \cup L_2 \cup L_3 \cup L_4) = \mathbf{Q}^2$$

for all rational surjective  $\varphi \in \text{Mat}_{2,4}(\mathbf{Q})$ .

Below  $X$  denotes a subvariety of  $\mathbf{G}_m^n$  with dimension  $r$  as in our theorem. We give a brief overview of the proof of Theorem 11 in the case  $s = r$ . The following steps do not reflect the formal layout of the argument as it is given. We rather present a simplified account which stays true to the general idea.

**Step I:** Suppose  $\varphi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  is a homomorphism of algebraic groups. We will see that  $\varphi$  can be represented as an  $n \times r$  matrix with integral coefficients. The morphism  $\varphi$  is surjective if and only if the associated matrix has rank  $r$  and we shall assume this. We let  $|\cdot|$  denote any fixed norm on the vector space of  $n \times r$ -matrices in real coefficients. The restriction  $\varphi|_X$  of  $\varphi$  to  $X$  is a morphism with a certain degree  $\Delta(\varphi) \geq 0$ .

In Section 6 we will prove a general height inequality on correspondences in  $\mathbf{P}^n \times \mathbf{P}^r$ . If we apply it to the Zariski closure of the graph of  $\varphi|_X$  in  $\mathbf{P}^n \times \mathbf{P}^r$  we obtain two positive constants  $c_1$  and  $c_2(\varphi) > 0$  with the following properties. First,  $c_1$  and  $c_2(\varphi)$  are completely explicit and given in terms of the height of  $X$ , the degree of  $X$ , and  $n$ . Actually, we allow  $c_2(\varphi)$  to depend on  $\varphi$ . Second, the lower bound

$$(2.2) \quad h(\varphi(p)) \geq c_1 |\varphi| \frac{\Delta(\varphi)}{|\varphi|^r} h(p) - c_2(\varphi)$$

holds for all algebraic points  $p$  in a Zariski open and dense subset of  $X$  (which depends on  $\varphi$ ). Moreover, the complement of said Zariski open set can be effectively determined. This means that we can bound the height and degree of each irreducible component as well as their number in terms of  $X$  and  $\varphi$ . Much of the work involved in deducing this inequality goes into making all estimates explicit.

We observe that if  $\varphi(p)$  is the unit element, then the left-hand side of (2.2) vanishes. This is will be the common situation as the kernel of  $\varphi$  is an algebraic subgroup of codimension  $r$ . Moreover, any algebraic subgroup of codimension  $r$  arises as such a kernel. So if  $p$  is inside the Zariski open subset and if  $\Delta(\varphi) > 0$  we obtain

$$h(p) \leq \frac{c_2(\varphi)}{c_1} \frac{1}{|\varphi|} \frac{|\varphi|^r}{\Delta(\varphi)}.$$

This is indeed a height upper bound. But it is not clear if it implies a uniform height upper bound: the right-hand side seems to depend heavily on  $\varphi$ . Even worse, the Zariski open set on which it holds also depends on  $\varphi$ .

**Step II:** In the second step we get a hold on the degree  $\Delta(\varphi)$  of  $\varphi|_X$ . This is an integer and it cannot be negative. It vanishes if and only if all fibers of  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$  have positive dimension. Each such fibers arises as the intersection of  $X$  with the translate of an algebraic subgroup of codimension  $r$ . From our definition we find that  $X$  is covered by anomalous subvarieties, so  $X^{\text{oa},[r]} = \emptyset$ . In this case, Theorem 11 is void. So we may assume without loss of generality that  $\Delta(\varphi) > 0$  for all surjective  $\varphi$ .

What we want to do next is to transform this non-vanishing statement into a sufficiently strong and explicit lower bound in terms of  $\varphi$ . This was also a key step in the first proof of the Bounded Height Conjecture. The main tool there was Ax's Theorem combined with a compactness argument. In the current approach we will still need Ax's Theorem. But the compactness argument is replaced by a result in Tropical Geometry in our Section 8. The tropical variety associated to  $X$  is a finite union of  $r$ -dimensional polyhedral cones. This data can be determined effectively in terms of  $X$ . The tropical variety associated to the kernel of  $\varphi$  is quite simply a vector subspace of  $\mathbf{Q}^n$  of dimension  $n - r$ . Determining  $\Delta(\varphi)$  essentially amounts to intersecting the tropical variety of  $X$  with this vector subspace and counting the points on the tropical intersection with correct multiplicities. This last feat is attained by the Theorem of Sturmfels and Tevelev. We will be able to bound  $\Delta(\varphi)$  from below using a certain polynomial in the entries of



$\varphi$ . Ax's Theorem will imply that this polynomial does not vanish when evaluated on *real*  $n \times r$  matrices of maximal rank. This information in conjunction with a compactness argument is enough to provide a qualitative lower bound for this polynomial. Using an explicit version of Łojasiewicz's inequality due to Rémond [40] we can transform this qualitative bound into a quantitative one. The overall effect will be

$$(2.3) \quad \Delta(\varphi) \geq c_3 |\varphi|^r$$

with  $c_3 > 0$  independent of  $\varphi$ , at least for  $\varphi$  not too close to a matrix of rank less than  $r$ . From this inequality we see that the factor  $|\varphi|^r / \Delta(\varphi)$  is harmless.

**Step III:** There remains the issue that  $\varphi$  depends on the algebraic subgroup of  $\mathbf{G}_m^n$ . Recall that  $c_2(\varphi)$  and the Zariski open set on which inequality (2.2) holds both depend on  $\varphi$ . In order to ensure that our height inequality remains valid for all possible algebraic subgroups we would have to restrict to the intersection of infinitely many Zariski open sets. This is not a wise idea. Luckily, this problem can be resolved using a rather simple trick found in Section 9. We relax the condition on  $\varphi$ . Instead of asking that  $\varphi(p)$  is the unit element and hence forcing the left-hand side of (2.2) to vanish, we instead only want  $h(\varphi(p))$  to be bounded above linearly in terms of  $h(p)$ . The constant involved will depend only on  $n$ . This can be achieved by replacing the matrix whose kernel defines the algebraic subgroup containing  $p$  by a sufficiently good approximation. The payoff is that we may choose  $\varphi$  from a fixed finite set depending only on  $X$ . So we will only need to take a finite intersection to apply (2.2). Moreover, the constant  $c_2(\varphi)$  in this inequality becomes harmless. The right-hand side of (2.2) involves  $|\varphi|^{r+1} / \Delta(\varphi)$ . Compared with (2.3) we get an extra factor  $|\varphi|$ . This factor will be used to defeat the factor in front of  $h(p)$  appearing in the upper bound for  $h(\varphi(p))$ .

Amalgamating step 1 through 3 yields the Generically Bounded Height Theorem stated below as Theorem 13. Roughly speaking, it states that if  $X^{\text{oa},[r]} \neq \emptyset$ , then there is a non-empty Zariski open subset  $U$  of  $X$  on which any point lying in  $(\mathbf{G}_m^n)^{[r]}$  has height bounded from above. The bound for the height of  $p$  is given explicitly and so are bounds for height, degree, and number of the irreducible components of  $X \setminus U$ .

**Step IV:** The fourth and final step deals with passing from the unspecified  $U$  to  $X^{\text{oa},[r]}$  itself. The idea here is to work with the irreducible components of  $X \setminus U$  and apply induction on the dimension. This step is carried out in Section 10. The cost of this descent argument becomes apparent when one compares the bounds in Theorem 13 and in our main result, Theorem 11.

The author would like to thank Eric Katz for pointing me towards Sturmfels and Tevelev's work. He is also grateful to Jenia Tevelev for answering tropically related questions.

### 3. NOTATION

**3.1. Generalities.** This section serves as a reference for general notation used throughout the whole article.

We use  $\mathbf{N}$  to denote the positive integers  $\{1, 2, 3, \dots\}$  and  $\mathbf{N}_0$  for the non-negative integers  $\mathbf{N} \cup \{0\}$ .

Let  $R$  be a commutative unitary ring and  $p = (p_1, \dots, p_n)$  a tuple of elements in  $R$ . If  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ , we set  $p^\alpha = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$  where our convention is  $0^0 = 1$ . For  $k \in \mathbf{N}_0$  we define  $p^k = (p_1^k, \dots, p_n^k)$ . If all  $p_i$  are in  $R^\times$ , the group of units of  $R$ , then we

allow  $\alpha$  to have negative entries and  $k \in \mathbf{Z}$ . It is convenient to set  $|\alpha|_1 = \alpha_0 + \cdots + \alpha_n$ . Say  $I$  is an ideal of  $R$ . If  $R$  is graded by the integers, then  $I_a$  will denote the homogeneous elements in  $I$  with degree  $a \in \mathbf{Z}$ . If  $R$  is bigraded by  $\mathbf{Z}^2$  then  $I_{(a,b)}$  denotes the elements in  $I$  with bidegree  $(a,b) \in \mathbf{Z}^2$ .

If  $A$  is any matrix, then  $A^\top$  is its transpose and  $\text{rk}(A)$  is its rank.

We fix  $\overline{\mathbf{Q}}$  to be the algebraic closure of  $\mathbf{Q}$  in  $\mathbf{C}$ , the field of complex numbers.

Throughout this paper, and if not stated otherwise,  $n, r \in \mathbf{N}$  are fixed integers and  $\mathbf{X} = (X_0, \dots, X_n)$ ,  $\mathbf{Y} = (Y_0, \dots, Y_r)$  are collections of independent variables. We often abbreviate  $\overline{\mathbf{Q}}[X_0, \dots, X_n]$  by  $\overline{\mathbf{Q}}[\mathbf{X}]$  and  $\overline{\mathbf{Q}}[Y_0, \dots, Y_r]$  by  $\overline{\mathbf{Q}}[\mathbf{Y}]$ . These rings have a natural grading and  $\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$  has a natural bigrading. In our notation we have  $\mathbf{X}^\alpha \in \overline{\mathbf{Q}}[\mathbf{X}]_{|\alpha|_1}$  for  $\alpha \in \mathbf{N}_0^{n+1}$ .

If  $\alpha$  is as before, then the multinomial coefficient  $\binom{|\alpha|_1}{\alpha}$  is  $\frac{|\alpha|_1!}{\alpha_0! \cdots \alpha_n!}$ .

For a rational prime  $p$  we let  $|\cdot|_p$  denote the  $p$ -adic absolute value on  $\mathbf{Q}$ . Let  $[x]$  denote the greatest integer at most  $x \in \mathbf{R}$ .

*Remark 3.1.* Say  $k = |\alpha|_1$ , it is well-known that

$$\begin{aligned} \log \left| \binom{k}{\alpha} \right|_p &= - \left( \sum_{\substack{e \geq 1 \\ p^e \leq k}} \left[ \frac{k}{p^e} \right] - \left[ \frac{\alpha_0}{p^e} \right] - \cdots - \left[ \frac{\alpha_n}{p^e} \right] \right) \log p \geq - \left( \sum_{\substack{e \geq 1 \\ p^e \leq k}} n + 1 \right) \log p \\ &= - \left[ \frac{\log k}{\log p} \right] (n + 1) \log p \geq -(n + 1) \log k. \end{aligned}$$

Thus we obtain the estimate

$$(3.1) \quad \left| \binom{k}{\alpha} \right|_p \geq \begin{cases} k^{-(n+1)} & : \text{if } p \leq k, \\ 1 & : \text{else wise.} \end{cases}$$

It will provide useful in many of our estimates.

Let for the moment  $K$  be an algebraically closed field contained in  $\mathbf{C}$ . We define an isomorphism of  $K$ -vector spaces

$$\iota : K[\mathbf{X}]_a \rightarrow K^{\binom{n+d}{d}}$$

by setting  $\iota(\mathbf{X}^i) = \binom{a}{i}^{-1/2} e_i$  for  $i \in \mathbf{N}_0^{n+1}$  with  $|i|_1 = a$ ; here  $(e_i)_i$  is the standard basis of  $K^{\binom{n+d}{d}}$  ordered lexicographically. We get an analog isomorphism  $\iota$  on  $K[\mathbf{Y}]_a$ . If  $b \in \mathbf{N}_0$ , these linear maps induce an isomorphism between  $K[\mathbf{X}, \mathbf{Y}]_{(a,b)}$  and the appropriate power  $K$ . They also induce isomorphisms between finite direct sums of  $K[\mathbf{X}, \mathbf{Y}]_{(a,b)}$  and the appropriate power of  $K$ . By abuse of notation, all these isomorphisms will be denoted by  $\iota$ .

Finally, we introduce some notation in connection with the algebraic torus  $\mathbf{G}_m^n$ .

A homomorphism  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  will always mean a homomorphism of algebraic groups. In particular, it sends the unit element to the unit element. Each homomorphism  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  is uniquely determined by a matrix  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$  with rows  $\varphi_1, \dots, \varphi_r \in \mathbf{Z}^n$ . Indeed, the homomorphism is given as  $x \mapsto (x^{\varphi_1}, \dots, x^{\varphi_r})$ . We will often use the same symbol  $\varphi$  for the matrix and corresponding homomorphism. Any algebraic subgroup of  $\mathbf{G}_m^n$  with codimension  $r$  is the kernel of some  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$  of rank  $r$ , cf. Corollary 3.2.15 [4].



We fix the open immersion  $\mathbf{G}_m^n \hookrightarrow \mathbf{P}^n$  given by  $(p_1, \dots, p_n) \mapsto [1 : p_1 : \dots : p_n]$ .

**3.2. Heights of an Algebraic Point.** We now spend some time defining the height of algebraic numbers and also algebraic points in projective space.

Let  $K$  be a number field. We let  $M_K$  denote the set of places of  $K$ . Let  $v \in M_K$ . We will identify  $v$  with an absolute value on  $K$  which when restricted to  $\mathbf{Q}$  is  $|\cdot|_p$  for some rational prime  $p$  or the complex absolute on  $\mathbf{Q}$ . We call  $v$  finite if it satisfies the ultrametric triangle inequality and we call  $v$  infinite if it does not.

If  $v$  is a finite place we let  $K_v$  denote a completion of  $K$  with respect to  $v$  and let  $\sigma_v$  be the embedding  $K \hookrightarrow K_v$ . We also fix  $\mathbf{C}_v$ , a completion of an algebraic closure of  $K_v$ , together with an embedding  $K \hookrightarrow \mathbf{C}_v$ , which by abuse of notation we also call  $\sigma_v$ . We write  $|\cdot|_v$  for the  $v$ -adic absolute value on  $\mathbf{C}_v$ .

An infinite  $v$  corresponds to an embedding of  $K$  into  $\mathbf{R}$  or  $\mathbf{C}$ ; in the latter case the embedding is only determined up-to complex conjugation. We take  $K_v = \mathbf{R}$  or  $K_v = \mathbf{C}$  and  $\sigma_v$ , accordingly. Moreover, we always set  $\mathbf{C}_v = \mathbf{C}$  in the infinite case. We sometimes abbreviate  $|\cdot| = |\cdot|_v$  for the standard absolute value on  $\mathbf{C}$ ;

By abuse of notation we let  $\mathbf{Q}_v$  denote  $\mathbf{Q}_{v|\mathbf{Q}}$ .

If  $p = (p_0, \dots, p_n) \in \mathbf{C}_v^{n+1}$ , we set

$$|p|_v = \begin{cases} (\sum_{i=0}^n |p_i|^2)^{1/2} & : \text{ if } v \text{ is infinite and} \\ \max_{0 \leq i \leq n} |p_i|_v & : \text{ if } v \text{ is finite.} \end{cases}$$

Now let us assume  $p = (p_0, \dots, p_n) \in K^{n+1} \setminus \{0\}$ . We define the height of  $p$  as

$$h(p) = \frac{1}{[K : \mathbf{Q}]} \sum_{v \in M_K} [K_v : \mathbf{Q}_v] \log |p|_v.$$

This height is invariant under multiplication of  $p$  by a non-zero scalar and under a finite field extension of  $K$ . The proof of these statements is similar as the argument for heights with the sup-norm at the infinite places given in Chapter 1.5 [4]. Hence we obtain a well-defined height function

$$h : \mathbf{P}^n(\overline{\mathbf{Q}}) \rightarrow [0, \infty).$$

Using the open immersion from Section 3.2 we obtain a height function  $\mathbf{G}_m^n(\overline{\mathbf{Q}}) \rightarrow [0, \infty)$ ; for which we use the same letter  $h$ . In addition to this height function with  $l^2$ -norm at infinite places we will use the height function

$$h_s : \mathbf{G}_m^n(\overline{\mathbf{Q}}) \rightarrow [0, \infty)$$

with the sup-norm at infinite places, cf. Chapter 1.5 [4]. This height has the advantage that it plays along nicely with the group law on  $\mathbf{G}_m^n(\overline{\mathbf{Q}})$ . For example, in the case  $n = 1$  and if  $p, q \in \mathbf{G}_m(\overline{\mathbf{Q}})$  then  $h_s(p^k) = |k| h_s(p)$  for all  $k \in \mathbf{Z}$  and  $h_s(pq) \leq h_s(p) + h_s(q)$ . Both statements follow from the definition of  $h_s$  and for the first one needs the product formula if  $k < 0$ . If  $p = (p_1, \dots, p_n) \in \mathbf{G}_m^n(\overline{\mathbf{Q}})$ , then

$$(3.2) \quad \max\{h_s(p_1), \dots, h_s(p_n)\} \leq h_s(p) \leq h_s(p_1) + \dots + h_s(p_n)$$

is also a consequence of the definition. We conclude that  $h_s(p^u) \leq n|u|_\infty h_s(p)$ . Finally,  $h_s(p) \leq h(p)$  because the sup-norm is always at most the  $l^2$ -norm and  $h(p) \leq \frac{1}{2} \log(n+1) + h_s(p)$  from a standard norm inequality. Moreover, if  $k \in \mathbf{N}_0$  then  $h_s(p^k) = k h_s(p)$ . In the following, these inequalities and statements will be referred to as basic height properties.

Say  $V$  is a finite direct sum of various  $\overline{\mathbf{Q}}[\mathbf{X}]_a$  and  $\overline{\mathbf{Q}}[\mathbf{Y}]_b$  and  $\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,b)}$  with  $a, b \in \mathbf{Z}$ . Then  $\iota$  as described above defines an isomorphism of  $V$  with some power of  $\overline{\mathbf{Q}}$ . If  $P \in V$  we define  $h(P) = h(\iota(P))$ . Moreover, if the coefficients of  $\iota(P)$  are in  $K$  we set  $|P|_v = |\iota(P)|_v$  for any place  $v$  of  $K$ . Then we obtain a height function, also denoted by  $h$ , defined on non-zero elements of  $V$ .

We also need a height function defined on  $\text{Mat}_{M,N}(\overline{\mathbf{Q}})$ , the vector space of  $M \times N$  matrices with algebraic coefficients. Let  $A \in \text{Mat}_{M,N}(K)$  where  $K$  is a finite extension of  $\mathbf{Q}$ . If  $1 \leq t \leq \text{rk}(A)$  is an integer and if  $v$  is an infinite place of  $K$  we define the local height of  $A$  as

$$h_{v,t}(A) = \frac{1}{2} \log \sum_{A' \subset A} |\det A'|_v^2$$

where the sum runs over all  $t \times t$  submatrices  $A'$  of  $A$ . If  $v$  is a finite place of  $K$  we define the local height as

$$h_{v,t}(A) = \log \max_{A' \subset A} |\det A'|_v$$

again the  $A'$  run over all  $t \times t$  submatrices of  $A$ . The height of  $A$  is then

$$h_t(A) = \frac{1}{[K : \mathbf{Q}]} \sum_v [K_v : \mathbf{Q}_v] h_{v,t}(A).$$

The following lemma provides a useful inequality concerning local heights of matrices.

**Lemma 3.1.** *Let  $A$  and  $B$  be as above and  $C$  a matrix with  $N$  rows and coefficients in  $K$ . Then  $h_{v,t}(AB) \leq h_{v,t}(A) + h_{v,t}(B)$  for all integers  $1 \leq t \leq \text{rk}(AB)$  and all places  $v$  of  $K$ .*

*Proof.* This is a direct consequence of the Cauchy-Binet formula together with the Cauchy-Schwarz inequality for infinite  $v$  and the ultrametric triangle inequality for finite  $v$ .  $\square$

We proceed by defining the height of an  $N$ -dimensional vector subspace  $V \subset \overline{\mathbf{Q}}^M$ . Indeed, let  $K$  be a finite extension of  $\mathbf{Q}$  and  $A \in \text{Mat}_{M,N}(K)$  a matrix whose column constitute a basis of  $V$ . We define  $h_{\text{Ar}}(V) = h_N(A)$ . Then  $h_{\text{Ar}}(V)$  is independent of the choice of basis, cf. Chapter 2.8 [4].

*Remark 3.2.* This height behaves well with certain constructions typical for vector spaces. If  $V \subset \mathbf{Q}^M$  and  $W \subset \overline{\mathbf{Q}}^{M'}$  are two vector subspaces. Then the subspace  $V \times W \subset \mathbf{Q}^{M+M'}$  satisfies

$$(3.3) \quad h_{\text{Ar}}(V \times W) \leq h_{\text{Ar}}(V) + h_{\text{Ar}}(W)$$

by Remark 2.8.9 [4].

If  $W \subset \overline{\mathbf{Q}}^M$ , then by a theorem of Schmidt we have

$$(3.4) \quad h_{\text{Ar}}(V \cap W) \leq h_{\text{Ar}}(V) + h_{\text{Ar}}(W),$$

cf. Theorem 2.8.13 [4].

Finally, by Proposition 2.8.10 [4] the height of a vector space equals the height of its orthogonal complement. In other words,

$$(3.5) \quad h_{\text{Ar}}(\{(u_1, \dots, u_M); \sum_i u_i v_i = 0 \text{ for all } (v_1, \dots, v_M) \in V\}) = h_{\text{Ar}}(V).$$

Using the notation introduced up until now we are able to define the height of a vector subspace  $V$  of finite direct sums of various  $\overline{\mathbf{Q}}[\mathbf{X}]_a$  and  $\overline{\mathbf{Q}}[\mathbf{Y}]_b$  and  $\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,b)}$ . Indeed, we set  $h_{\text{Ar}}(V) = h_{\text{Ar}}(\iota(V))$ .

Let  $X \subset \mathbf{P}^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  with homogeneous ideal  $I \subset \overline{\mathbf{Q}}[\mathbf{X}]$ . The arithmetic and geometric Hilbert functions of  $X$  are defined as

$$\mathcal{H}_a(a; X) = h_{\text{Ar}}(I_a) \quad \text{resp.} \quad \mathcal{H}_g(a; X) = \dim \overline{\mathbf{Q}}[\mathbf{X}]_a / I_a,$$

respectively.

Let  $Z \subset \mathbf{P}^n \times \mathbf{P}^m$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  with bihomogeneous ideal  $I \subset \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$ . The arithmetic and geometric Hilbert functions of  $Z$  are defined as

$$\mathcal{H}_a(a, b; Z) = h_{\text{Ar}}(I_{(a,b)}) \quad \text{and} \quad \mathcal{H}_g(a, b; Z) = \dim \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,b)} / I_{(a,b)},$$

respectively.

**3.3. Height of a Projective Variety.** We will briefly describe the definition of the height of an irreducible closed subvariety  $X \subset \mathbf{P}^n$  defined over  $\overline{\mathbf{Q}}$  as it is given by Philippon [31].

In this section we use  $\mathbf{V}_i$  to denote an  $N_i = \binom{n+d_i}{d_i}$ -tuple of independent variables  $U_{i0}, \dots, U_{i,N_i-1}$ . Each packet  $\mathbf{V}_i$  corresponds to coefficients of a homogeneous polynomial in  $n+1$  variables of degree  $d_i$ .

Let  $f \in K[\mathbf{V}_0, \dots, \mathbf{V}_r] \setminus \{0\}$  be homogeneous of degree  $\deg_{\mathbf{V}_i}(f)$  in the variables  $\mathbf{V}_i$  for  $0 \leq i \leq r$ .

If  $v$  is a finite place of  $K$  we define  $m_v(f)$  to be the logarithm of maximum of the absolute values of coefficients of  $f$  with respect to  $v$ .

Say  $v$  is an infinite place of  $K$  and  $\sigma = \sigma_v : K \rightarrow \mathbf{C}$  a corresponding embedding. We set

$$m_v(f) = \int_{S_{N_0} \times \dots \times S_{N_r}} \log |\sigma(f)(u_0, \dots, u_r)| d\mu_0(u_0) \cdots d\mu_r(u_r) + \sum_{i=0}^r \deg_{\mathbf{V}_i}(f) \sum_{j=1}^{N_i-1} \frac{1}{2^j}$$

where  $S_{N_i}$  is the unit circle in  $\mathbf{C}^{N_i}$  on which  $\mu_i$  is the measure, invariant under the unitary group, of total mass 1.

*Remark 3.3.* It is known that

$$(3.6) \quad \sum_{v \in M_K} [K_v : \mathbf{Q}_v] m_v(f) \geq 0.$$

Let us indicate the proof.

The left-hand side of (3.6) remains unchanged when replacing  $K$  by a larger number field. By the product formula, cf. Chapter 1.4 [4], it also remains unchanged when replacing  $f$  by a multiple with factor in  $K^\times$ . After replacing  $K$  by a larger number field we may find  $\lambda \in K^\times$  such that  $\log |\lambda|_v + m_v(f) = 0$  for all finite places  $v$ . We replace  $f$  by  $\lambda f$ , which has coefficients in the ring of integers of  $K$ . It suffices to show that  $\sum_{v \text{ infinite}} [K_v : \mathbf{Q}_v] m_v(f) \geq 0$ . So say  $v$  is an infinite place of  $K$ . By Lelong's Théorème 4 [26] the value  $m_v(f)$  is at least the logarithmic Mahler measure of  $\sigma_v(f)$ . Because the Mahler measure is multiplicative,  $\sum_{v \text{ infinite}} [K_v : \mathbf{Q}_v] m_v(f)$  is at least the logarithmic Mahler measure of  $\prod_{v \text{ infinite}} \sigma_v(f)^{[K_v : \mathbf{Q}_v]}$ . This is a non-zero polynomial in

integer coefficients. But the logarithmic Mahler measure of a non-zero polynomial in integer coefficients is non-negative. Our claim (3.6) follows.

Let  $r = \dim X$  and let  $K$  be a number field over which  $X$  is defined. Say  $d = (d_0, \dots, d_r) \in \mathbf{N}^{r+1}$ .

We let  $f_{X,d}$  denote the elimination or Chow form of  $X$  associated to  $d$ . For a more proper survey on Chow forms and their the properties stated below we refer to Philippon's article [33]. It is uniquely defined up-to multiplication by a scalar in  $K^\times$ . The Chow form is homogeneous in each  $\mathbf{V}_i$  of degree  $\deg(X)d_0 \cdots d_r/d_i$ .

Now we are ready to define the height of  $X$ . It is

$$(3.7) \quad h(X) = \frac{1}{d_0 \cdots d_r} \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} m_v(f_{X,d}).$$

This value is independent of  $d_0, \dots, d_r$ , the choice of number field  $K$ , and the choice of  $f_{X,d}$ . This height is sometimes called the Faltings height.

From (3.6) we deduce  $h(X) \geq 0$ .

Recall that we have fixed an open immersion  $\mathbf{G}_m^n \hookrightarrow \mathbf{P}^n$ . If  $X \subset \mathbf{G}_m^n$  is an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$ , then  $\deg(X)$  and  $h(X)$  are defined to be the degree and height of the Zariski closure of  $X$  in  $\mathbf{P}^n$ , respectively. Similarly, the height of a point in  $\mathbf{G}_m^n(\overline{\mathbf{Q}})$  is the height of its image in  $\mathbf{P}^n(\overline{\mathbf{Q}})$ .

#### 4. ESTIMATES FOR HILBERT FUNCTIONS

**4.1. The Geometric Hilbert Function.** Let  $Z$  be an irreducible closed subvariety of  $\mathbf{P}^n \times \mathbf{P}^r$ . The purpose of this section is to estimate  $\mathcal{H}_g(a, b; Z)$  explicitly in terms of  $H(a, b; Z)$ . The upper bound for the geometric Hilbert function in terms of the Hilbert polynomial given in Lemma 4.3 follows arguments of Bertrand and Kollár [2].

It is known that if  $\min\{a, b\}$  is large enough, then  $\mathcal{H}_g(a, b; Z)$  is the value at  $(a, b)$  of a polynomial depending only on  $Z$ . This uniquely determined polynomial is called the Hilbert polynomial of  $Z$ . We define  $H(T_1, T_2; Z) \in \mathbf{Q}[T_1, T_2]$  to be  $(\dim Z)!$  times the highest degree homogeneous part of the Hilbert polynomial. Section 3 [29] contains a treatment of Hilbert polynomials.

An treatment of the intersection theory needed in this article can be found in Fulton's book [17].

Let  $\pi_1 : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^n$  and  $\pi_2 : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^r$  denote the projection onto the two factors. Let  $\mathcal{O}(1)$  denote the dual of the tautological line bundle on any projective space. Then

$$(4.1) \quad H(T_1, T_2; Z) = \sum_{i=0}^{\dim Z} \binom{\dim Z}{i} (\pi_1^* \mathcal{O}(1)^i \pi_2^* \mathcal{O}(1)^{\dim Z - i} [Z]) T_1^i T_2^{\dim Z - i}.$$

where  $[Z]$  is the class of cycles on  $\mathbf{P}^n \times \mathbf{P}^r$  which are linearly equivalent to  $Z$ .

Let  $\mathbf{U}$  be an  $(n+1)(r+1)$ -tuple of independent variables  $(U_{ij})$  where  $0 \leq i \leq n$  and  $0 \leq j \leq r$ . For any field  $K$  we define the Segre homomorphism

$$s^* : K[\mathbf{U}] \rightarrow K[\mathbf{X}, \mathbf{Y}]$$

by setting  $s^*(U_{ij}) = X_i Y_j$ . By abuse of notation we also let  $s^*$  denote the restriction  $K[\mathbf{U}]_k \rightarrow K[\mathbf{X}, \mathbf{Y}]_{(k,k)}$  for all  $k \in \mathbf{N}_0$ .

The Segre homomorphism defines a morphism  $s : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^{nr+n+r}$  of varieties. We define

$$\deg(Z) = \deg(s(Z)) \quad \text{and} \quad h(Z) = h(s(Z)) \quad \text{if } Z \text{ is defined over } \overline{\mathbf{Q}}$$

where the degree of a subvariety of projective space is the degree of the cycle class obtained by intersecting the variety the appropriate number of times with  $\mathcal{O}(1)$ .

By abuse of notation,  $s^*$  will also denote the induced homomorphism between the Picard groups of  $\mathbf{P}^{nr+n+r}$  and  $\mathbf{P}^n \times \mathbf{P}^r$ .

Next we define the Veronese maps. Let  $a \in \mathbf{N}$  and let  $\tilde{\mathbf{X}}$  be the  $\binom{n+a}{a}$ -tuple  $(\tilde{X}_\gamma)$  where  $\gamma$  runs through all vectors in  $\mathbf{N}_0^{n+1}$  with  $|\gamma|_1 = a$ . We define the twisted Veronese homomorphism

$$v_a^* : K[\tilde{\mathbf{X}}] \rightarrow K[\mathbf{X}]$$

by setting  $v_a^*(\tilde{X}_\gamma) = \binom{a}{\gamma}^{1/2} \mathbf{X}^\gamma$ ; of course,  $v_a^*$  is only defined if  $K$  contains all roots  $\binom{a}{\gamma}^{1/2}$ . The choice of the square root will be irrelevant in our applications. By abuse of notation we also let  $v_a^*$  denote the restriction  $K[\tilde{\mathbf{X}}]_k \rightarrow K[\mathbf{X}]_{ak}$ .

The Veronese homomorphism  $v_a^*$  induces a closed immersion  $v_a : \mathbf{P}^n \rightarrow \mathbf{P}^{\binom{n+a}{a}-1}$  of varieties. If  $b \in \mathbf{N}$ , then the product  $v_{ab} : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^{\binom{n+a}{a}-1} \times \mathbf{P}^{\binom{r+b}{b}-1}$  is also a closed immersion.

By abuse of notation,  $v_a^*$  and  $v_{ab}^*$  will also denote the induced homomorphisms between Picard groups of  $\mathbf{P}^{\binom{n+a}{a}-1} \times \mathbf{P}^{\binom{r+b}{b}-1}$  and of  $\mathbf{P}^n \times \mathbf{P}^r$ .

**Lemma 4.1.** *We have  $H(1, 1; Z) = \deg(Z)$  and  $H(a, b; Z) \leq \max\{a, b\}^{\dim Z} \deg(Z)$  for  $a, b \in \mathbf{N}$ .*

*Proof.* By definition we have  $\deg(s(Z)) = (\mathcal{O}(1)^{\dim Z}[s(Z)])$ . The projection formula, and the fact that  $s$  is a closed embedding imply  $\deg(s(Z)) = (s^*\mathcal{O}(1)^{\dim Z}[Z])$ . Since  $s^*\mathcal{O}(1) = \pi_1^*\mathcal{O}(1) \otimes \pi_2^*\mathcal{O}(1)$  we deduce

$$\deg(s(Z)) = \sum_{i+j=\dim Z} \binom{\dim Z}{i} (\pi_1^*\mathcal{O}(1)^i \pi_2^*\mathcal{O}(1)^j[Z]) = H(1, 1; Z).$$

The first statement of the lemma follows.

The second statement follows from the first one and  $(\pi_1^*\mathcal{O}(1)^i \pi_2^*\mathcal{O}(1)^j[Z]) \geq 0$ .  $\square$

**Lemma 4.2.** *We have  $H(1, 1; v_{ab}(Z)) = H(a, b; Z)$  for  $a, b \in \mathbf{N}$ .*

*Proof.* For this lemma we write  $\pi'_1$  and  $\pi'_2$  for the first and second projection on  $\mathbf{P}^{\binom{n+a}{a}-1} \times \mathbf{P}^{\binom{r+b}{b}-1}$ , respectively. By definition and since  $[v_{ab}(Z)] = v_{ab*}([Z])$  we have

$$\begin{aligned} H(1, 1; v_{ab}(Z)) &= \sum_{i+j=\dim Z} \binom{\dim Z}{i} (\pi'_1{}^*\mathcal{O}(1)^i \pi'_2{}^*\mathcal{O}(1)^j v_{ab*}[Z]) \\ &= \sum_{i+j=\dim Z} \binom{\dim Z}{i} (v_{ab}^* \pi'_1{}^* \mathcal{O}(1)^i v_{ab}^* \pi'_2{}^* \mathcal{O}(1)^j [Z]) \end{aligned}$$

where we used the projection formula. We note  $\pi'_1 \circ v_{ab} = v_a \circ \pi_1$  and  $v_{ab}^* \pi'_1{}^* \mathcal{O}(1) = (\pi'_1 \circ v_{ab})^* \mathcal{O}(1) = (v_a \circ \pi_1)^* \mathcal{O}(1) = \pi_1^* v_a^* \mathcal{O}(1) = \pi_1^* \mathcal{O}(1)^{\otimes a}$ . Similarly,  $v_{ab}^* \pi'_2{}^* \mathcal{O}(1) = \pi_2^* \mathcal{O}(1)^{\otimes b}$ . The lemma follows.  $\square$

**Lemma 4.3.** *Assume  $Z$  is a curve, then  $\mathcal{H}_g(a, b; Z) \leq 1 + H(a, b; Z)$  for  $a, b \in \mathbf{N}$ .*

*Proof.* Lemma 1.1 [2] implies  $\mathcal{H}_g(a, b; Z) - 1 \leq \deg(s(v_{ab}(Z)))$ . So  $\mathcal{H}_g(a, b; Z) \leq 1 + H(1, 1; v_{ab}(Z))$  by Lemma 4.1. We conclude the current lemma by referring to Lemma 4.2.  $\square$

We now generalize this bound to higher dimension using a Bertini-type argument.

**Lemma 4.4.** *Say  $\dim Z \geq 1$ . We have*

$$\mathcal{H}_g(ak, bk; Z) \leq H(a, b; Z) \binom{\dim Z + k}{\dim Z}$$

for  $a, b, k \in \mathbf{N}$ .

*Proof.* We prove the lemma by induction on  $d = \dim Z$ . If  $d = 1$ , then Lemma 4.3 leads to  $\mathcal{H}_g(ak, bk; Z) \leq 1 + H(ak, bk; Z) = 1 + H(a, b; Z)k$  because the Hilbert polynomial is homogeneous of degree  $d$ . But  $H(a, b; Z) \geq 1$  because it is a positive integer by (4.1). Hence  $\mathcal{H}_g(ak, bk; Z) \leq H(a, b; Z)(1 + k) = H(a, b; Z) \binom{1+k}{1}$  as desired. So let us assume  $d \geq 2$ .

Let  $I \subset R = \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$  be the bihomogeneous prime ideal of  $Z$ . By Bertini's Theorem, c.f. Corollaire 6.11 (2) and (3), page 89 [25] there is  $F \in R_{(a,b)} \setminus I_{(a,b)}$  such that  $I + F \cdot R$  is again a prime ideal of a variety  $Z' \subset \mathbf{P}^n \times \mathbf{P}^r$  of dimension  $d - 1$ . If  $k \geq 1$ , then

$$0 \rightarrow R_{(a(k-1), b(k-1))} / I_{(a(k-1), b(k-1))} \rightarrow R_{(ak, bk)} / I_{(ak, bk)} \rightarrow R_{(ak, bk)} / (I + F \cdot R)_{(ak, bk)} \rightarrow 0$$

is an exact sequence, the second arrow being multiplication by  $F$ . So  $\mathcal{H}_g(ak, bk; Z) = \mathcal{H}_g(a(k-1), b(k-1); Z) + \mathcal{H}_g(ak, bk; Z')$ . By induction on  $k$  we have

$$\mathcal{H}_g(ak, bk; Z) = \sum_{j=0}^k \mathcal{H}_g(a, b; Z').$$

We have  $\dim Z' = d - 1$  and by induction  $\mathcal{H}_g(a, b; Z') \leq H(a, b; Z') \binom{d-1+j}{d-1}$  if  $j \neq 0$ . This upper bound also holds for  $j = 0$  since  $\mathcal{H}_g(0, 0; Z') = 1$ . So

$$\mathcal{H}_g(ak, bk; Z) \leq H(a, b; Z') \sum_{j=0}^k \binom{d-1+j}{d-1} = H(a, b; Z') \binom{d+k}{d}$$

by properties to binomial coefficients.

Philippon's Lemme 3.1 [29] implies  $H(a, b; Z') = H(a, b; Z)$  and this completes the proof.  $\square$

Now we come to lower bounds.

**Lemma 4.5.** *Let  $\Delta = (\pi_2^* \mathcal{O}(1)^{\dim Z} [Z])$ . We have*

$$\mathcal{H}_g(a, b; Z) \geq \Delta \binom{\dim Z + b - \Delta}{\dim Z}$$

for all  $a, b \in \mathbf{N}$  with  $a \geq \Delta$  and  $b \geq \Delta$ .



*Proof.* Since  $\Delta$  is a non-negative integer we may assume  $\Delta \geq 1$ . For brevity, we write  $d = \dim Z$ . By Bertini's Theorem we find linear forms  $l_0, \dots, l_d \in \overline{\mathbf{Q}}[\mathbf{Y}]$  which define a rational map  $[l_0 : \dots : l_d] : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^d$  whose restriction  $\Psi$  to  $Z$  is a morphism of degree  $\Delta$ . We may suppose that the form  $l_0$  does not vanish identically on  $Z$ . The morphism  $\Psi$  induces an extension of function fields  $\overline{\mathbf{Q}}(Z)/\Psi^*\overline{\mathbf{Q}}(\mathbf{P}^d)$  of degree  $\Delta$ .

We shall assume that the projective coordinates  $X_0$  and  $Y_0$  do not vanish identically on  $Z$ . Otherwise, the argument given below goes through when working with some other pair  $X_i$  and  $Y_j$ .

We apply the Primitive Element Theorem to find a rational linear combination  $\gamma$  of  $X_1/X_0, \dots, X_n/X_0, Y_1/Y_0, \dots, Y_r/Y_0$  which, when considered as an element of  $\overline{\mathbf{Q}}(Z)$ , generates the field extension  $\overline{\mathbf{Q}}(Z)/\Psi^*\overline{\mathbf{Q}}(\mathbf{P}^d)$ .

We set  $b' = b - \Delta \geq 0$  and let  $P_0, \dots, P_{\Delta-1} \in \overline{\mathbf{Q}}[Y_0, \dots, Y_j]_{b'}$ . Let us consider

$$(4.2) \quad \left( \sum_{k=0}^{\Delta-1} \gamma^k P_k \left( 1, \frac{l_1}{l_0}, \dots, \frac{l_d}{l_0} \right) \right) X_0^a Y_0^\Delta l_0^{b'}$$

as an element of  $\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a, \Delta+b')}$ . But the sum in brackets is a rational function on  $Z$ . Since  $\gamma$  has degree  $\Delta$  over  $\Psi^*\overline{\mathbf{Q}}(\mathbf{P}^i \times \mathbf{P}^j)$  we see that (4.2) vanishes on  $Z$  if and only if all  $P_k$  are zero. The current lemma follows from this and  $\dim \overline{\mathbf{Q}}[Y_0, \dots, Y_d]_{b'} = \binom{d+b-\Delta}{d}$ .  $\square$

**4.2. The Arithmetic Hilbert Function.** In this subsection we bound from above the arithmetic Hilbert function of an irreducible closed subvariety  $X \subset \mathbf{P}^n$  defined over  $\overline{\mathbf{Q}}$ .

We begin by citing a result of David and Philippon [15].

Say  $V$  is a subvariety of projective space defined over a field  $K$ . If  $\sigma : K \rightarrow L$  is an embedding of  $K$  into another field  $L$ , then  $V_\sigma$  denotes the induced subvariety of projective space defined over  $L$ .

**Lemma 4.6.** *Let  $V$  be a non-trivial vector subspace of  $\overline{\mathbf{Q}}^N$  and let  $\tilde{V} \subset \mathbf{P}^{N-1}$  be the set of lines in  $V$ . If  $\epsilon > 0$  there is a linear form  $l$  on  $\overline{\mathbf{Q}}^N$  which does not vanish completely on  $V$  such that*

$$\sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \sup_{p \in \tilde{V}_{\sigma_v}(\mathbf{C}_v)} \frac{|\sigma(l)(p)|_v}{|p|_v} \leq -\frac{h_{\text{Ar}}(V)}{\dim V} + \frac{1}{\dim V} \sum_{i=1}^{\dim V-1} \sum_{j=1}^i \frac{1}{2j} + \epsilon$$

where  $K$  is a number field containing the coefficients of  $l$ . We remark that the quotient  $|\sigma(l)(p)|_v/|p|_v$  is well-defined for  $p \in \mathbf{P}^{N-1}(\mathbf{C}_v)$ .

*Proof.* This is Corollaire 4.9 [15] in the case  $R = \overline{\mathbf{Q}}$ .  $\square$

We now bound the arithmetic Hilbert function from above explicitly in terms of the height and degree of a variety. We essentially follow the argumentation given by Philippon in the proof of Théorème 7 [32]. Our inequality is absolute in the sense that the bound is independent of a field of definition.

**Proposition 4.1.** *If  $k \in \mathbf{N}$ , then*

$$\mathcal{H}_a(k; X) \leq \mathcal{H}_g(k; X) \left( k \frac{h(X)}{\deg(X)} + \frac{1}{2} \log \mathcal{H}_g(k; X) \right).$$

*Proof.* By definition we have  $\mathcal{H}_a(k; X) = h_{\text{Ar}}(I_k)$  where  $I \subset \overline{\mathbf{Q}}[X_0, \dots, X_n]$  is the ideal of  $X$ .

Say  $N = \binom{n+k}{k}$ , this is the maximal number of monomials in a homogeneous polynomial of degree  $k$  with  $n+1$  variables. We will apply Lemma 4.6 to

$$V = \{v \in \overline{\mathbf{Q}}^N; v^\top \cdot \iota(P) = 0 \text{ for all } P \in I_k\} \subset \overline{\mathbf{Q}}^N.$$

Equality (3.5) implies that the height of  $V$  is  $\mathcal{H}_a(k; X)$ . We have  $\dim V = \mathcal{H}_g(k; X) \geq 1$ .

Say  $\epsilon > 0$ . By Lemma 4.6 there is a linear form  $l$ , which we may identify with an element of  $\overline{\mathbf{Q}}^N$ , that does not vanish identically on  $V$  with

$$\sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \sup_{p \in \tilde{V}_{\sigma_v}(\mathbf{C}_v)} \frac{|\sigma_v(l)(p)|_v}{|p|_v} \leq -\frac{\mathcal{H}_a(k; X)}{\mathcal{H}_g(k; X)} + \frac{1}{\mathcal{H}_g(k; X)} \sum_{i=1}^{\mathcal{H}_g(k; X)-1} \sum_{j=1}^i \frac{1}{2j} + \epsilon$$

for a number field  $K \subset \overline{\mathbf{Q}}$  containing the coefficients of  $l$  and all of the finitely many algebraic numbers appearing in this proof.

There is a homogeneous  $P \in \overline{\mathbf{Q}}[\mathbf{X}]$  of degree  $k$  with  $l = \iota(P)$ . We write  $P = \sum_{|\lambda|_1=k} P_\lambda \mathbf{X}^\lambda$ .

Let  $f = f_{X,(1,\dots,1,k)} \in K[\mathbf{V}_0, \dots, \mathbf{V}_r]$  be a Chow form of  $X$ . In the notation of Section 3.3,  $\mathbf{V}_r$  is an  $N$ -tuple of variables corresponding each to a monomial of degree  $k$  in  $n+1$  variables. Let  $\rho(f) \in K[\mathbf{V}_0, \dots, \mathbf{V}_{r-1}]$  be the form obtained by specializing the variables  $\mathbf{V}_r$  to equal the corresponding coefficients of  $P$ .

Let  $v \in M_K$  and say  $\sigma = \sigma_v : K \rightarrow \mathbf{C}_v$  is the embedding choosen in Subsection 3.2.

We assume first that  $v$  is infinite. By Lemma 4.1 [33] there is a measure  $\Omega$  of total mass  $\deg(X)$  on  $X_\sigma(\mathbf{C})$  with

$$(4.4) \quad m_v(\rho(f)) - m_v(f) = \int_{X_\sigma(\mathbf{C})} \log \frac{|\sigma(P)(p)|}{|p|^k} \Omega(p).$$

Let  $p = (p_0, \dots, p_n) \in \mathbf{C}^{n+1}$  such that  $[p_0 : \dots : p_n] \in X_\sigma(\mathbf{C})$ . For  $\lambda \in \mathbf{N}_0^{n+1}$  with  $|\lambda|_1 = k$  we set  $q_\lambda = \sigma\left(\binom{k}{\lambda}^{1/2}\right)p^\lambda$  and  $q = (q_\lambda)_\lambda \in \mathbf{C}^N$ . Then  $q \in \sigma(V)$ . Moreover, we have

$$\sigma(P)(p) = \sum_{|\lambda|_1=k} \sigma(P_\lambda) p^\lambda = \sum_{|\lambda|_1=k} \sigma(P_\lambda) \sigma\left(\left(\binom{k}{\lambda}\right)^{-1/2}\right) q_\lambda = \sigma(\iota(P)) \cdot q^\top = \sigma(l)(q).$$

We evaluate

$$|p|^{2k} = \sum_{|\lambda|_1=k} \binom{k}{\lambda} |p^\lambda|^2 = \sum_{|\lambda|_1=k} |q_\lambda|^2 = |q|^2$$

and thus conclude

$$\frac{|\sigma(P)(p)|}{|p|^k} = \frac{|\sigma(l)(q)|}{|q|} \leq \sup_{q' \in \sigma(\tilde{V})(\mathbf{C})} \frac{|\sigma(l)(q')|}{|q'|}$$

Using (4.4) we obtain

$$(4.5) \quad m_v(\rho(f)) - m_v(f) \leq \deg(X) \log \sup_{q' \in \sigma(\tilde{V})(\mathbf{C})} \frac{|\sigma(l)(q')|}{|q'|}.$$

Now we assume that  $v$  is finite. By Lemme 2 [30] we have

$$(4.6) \quad m_v(\rho(f)) - m_v(f) = \int_{X_\sigma(\mathbf{C}_v)} \log \frac{|\sigma(P)(p)|_v}{|p|_v^k} \Omega_v(p),$$

with a measure  $\Omega_v$  on  $X_\sigma(\mathbf{C}_v)$  of mass  $\deg(X)$ .

Say  $p = (p_0, \dots, p_n) \in \mathbf{C}_v^{n+1}$  with  $[p_0 : \dots : p_n] \in X_\sigma(\mathbf{C}_v)$ . As in the case where  $v$  is infinite we have  $\sigma(P)(p) = \sigma(l)(q)$  where  $q \in \mathbf{C}_v^N$  is defined in a similar manner. This time  $|q|_v = \max_\lambda \{|q_\lambda|_v\} \leq \max_\lambda \{|p^\lambda|_v\} = \max\{|p_0|_v, \dots, |p_n|_v\}^k = |p|_v^k$ , so

$$\frac{|\sigma(P)(p)|_v}{|p|_v^k} \leq \frac{|\sigma(l)(q)|_v}{|q|_v}.$$

By (4.6) we deduce

$$(4.7) \quad m_v(\rho(f)) - m_v(f) \leq \deg(X) \log \sup_{y \in \tilde{V}(\mathbf{C}_v)} \frac{|\sigma(l)(y)|_v}{|y|_v}.$$

We multiply (4.5) and (4.7) with  $[K_v : \mathbf{Q}_v]/[K : \mathbf{Q}]$  and take the sum over all  $v \in M_K$ . Using the definition of  $h(X)$  given in (3.7) we obtain

$$-kh(X) + \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} m_v(\rho(f)) \leq \deg(X) \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \sup_{q \in \sigma_v(\tilde{V})(\mathbf{C}_v)} \frac{|\sigma(l)(q)|_v}{|q|_v}.$$

Recall that  $\sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} m_v(\rho(f)) \geq 0$  by (3.6). Using (4.3) we bound the right-hand side to get

$$(4.8) \quad \deg(X) \frac{\mathcal{H}_a(k; X)}{\mathcal{H}_g(k; X)} \leq kh(X) + \left( \frac{1}{\mathcal{H}_g(k; X)} \sum_{i=1}^{\mathcal{H}_g(k; X)-1} \sum_{j=1}^i \frac{1}{2j} + \epsilon \right) \deg(X).$$

If  $T \geq 1$  is an integer, then elementary inequalities lead to

$$\sum_{i=1}^{T-1} \sum_{j=1}^i \frac{1}{j} \leq \sum_{i=1}^{T-1} (1 + \log i) \leq T - 1 + \int_1^T \log(i) di = T \log T.$$

From (4.8) we conclude

$$\mathcal{H}_a(k; X) \leq k \mathcal{H}_g(k; X) \frac{h(X)}{\deg(X)} + \frac{1}{2} \mathcal{H}_g(k; X) \log \mathcal{H}_g(k; X) + \epsilon \mathcal{H}_g(k; X).$$

The proposition follows since  $\epsilon > 0$  was arbitrary.  $\square$

## 5. MORE ON SEGRE AND VERONESE

This section is on height inequalities in connection with Segre and Veronese morphisms. Both were defined in Section 4.1.

Let  $K$  be a field equipped with an automorphism  $\tau : K \rightarrow K$  of order at most 2. Let  $V$  be a finite dimensional vector space over  $K$ . A function  $\langle \cdot, \cdot \rangle : V \times V \rightarrow K$  is called a  $\tau$ -inner product, or short inner product, if it satisfies the following three properties.

- (i) It is  $K$ -linear in its first variable.
- (ii) We have  $\langle v, w \rangle = \tau(\langle w, v \rangle)$  for all  $w, v \in V$ .
- (iii) If  $v \in V$  and  $\langle v, w \rangle = 0$  for all  $w \in V$ , then  $v = 0$ .

If  $W$  is a vector subspace of  $V$ , then  $W^\perp$  is the orthogonal complement of  $W$ , i.e.  $W^\perp = \{v \in V; \langle v, w \rangle = 0 \text{ for all } w \in W\}$ . Property (iii) in the definition implies  $\dim W + \dim W^\perp = \dim V$ .

Now assume  $W$  is a second finite dimensional vector space over  $K$  with a  $\tau$ -inner product  $\langle \cdot, \cdot \rangle'$ . We call a linear map  $f : V \rightarrow W$  an isometry if

$$(5.1) \quad \langle f(v), f(v') \rangle' = \langle v, v' \rangle \quad \text{for all } v \in (\ker f)^\perp \quad \text{and } v' \in V.$$

If  $f$  is injective, then our notion of isometry is the usual one.

*Remark 5.1.* We claim that  $\ker f \cap (\ker f)^\perp = 0$ . Indeed, if  $v$  lies in this intersection, then  $0 = \langle 0, f(v') \rangle = \langle f(v), f(v') \rangle' = \langle v, v' \rangle$  for all  $v' \in V$ . So  $v = 0$  because of condition (iii) in the definition of  $\langle \cdot, \cdot \rangle$ .

On  $V \otimes W$  we have a natural  $\tau$ -inner product determined by  $\langle v \otimes v', w \otimes w' \rangle'' = \langle v, w \rangle \langle v', w' \rangle'$ .

To ease notation we sometimes use  $\langle \cdot, \cdot \rangle$  to denote inner-products on multiple vector spaces.

The next lemma is a simple application of linear algebra.

**Lemma 5.1.** *Let  $K$  and  $\tau$  be as above and assume  $V, W, V'$ , and  $W'$  are finite dimensional vector spaces over  $K$ , each one equipped with a  $\tau$ -inner product. If  $f : V \rightarrow W$  and  $f' : V' \rightarrow W'$  are isometries, then so is  $f \otimes f' : V \otimes V' \rightarrow W \otimes W'$ .*

*Proof.* We write  $\langle \cdot, \cdot \rangle$  for any inner product in this proof. The lemma formally follows from  $(\ker f)^\perp \otimes (\ker f')^\perp = (\ker f \otimes f')^\perp$  which we now show. To prove the inclusion “ $\subset$ ” we let  $v \in (\ker f)^\perp$ ,  $v' \in (\ker f')^\perp$ , and  $u = \sum_i u_i \otimes u'_i \in \ker f \otimes f'$ . So  $\langle v \otimes v', u \rangle = \sum_i \langle v \otimes v', u_i \otimes u'_i \rangle = \sum_i \langle v, u_i \rangle \langle v', u'_i \rangle$ . But  $f$  and  $f'$  are isometries, therefore  $\langle v \otimes v', u \rangle = \sum_i \langle f(v) \otimes f'(v'), f(u_i) \otimes f'(u'_i) \rangle = \langle f(v) \otimes f'(v'), (f \otimes f')(u) \rangle = 0$ . It follows that  $v \otimes v' \in (\ker f \otimes f')^\perp$  since  $u$  was arbitrary. The desired inclusion holds.

Now  $\dim(\ker f)^\perp \otimes (\ker f')^\perp = (\dim V - \dim \ker f)(\dim V' - \dim \ker f')$ . Moreover,  $\dim(\ker f \otimes f')^\perp = \dim V \otimes V' - \dim \ker f \otimes f' = \dim f(V) \otimes f'(V') = \dim f(V) \dim f'(V')$ . We conclude that  $(\ker f)^\perp \otimes (\ker f')^\perp$  and  $(\ker f \otimes f')^\perp$  have equal dimension, so they coincide.  $\square$

*Remark 5.2.* Let  $K$  be a field of characteristic 0 together with an involution  $\tau$ . We shall assume that if  $k \in \mathbf{N}$  and  $\sqrt{k} \in K$  then  $\tau(\sqrt{k}) = \sqrt{k}$ . We fix  $a \in \mathbf{N}_0$ . An important example of a  $\tau$ -inner product on  $K[\mathbf{X}]_a$  is defined in the following manner. If  $P = \sum_\gamma P_\gamma \mathbf{X}^\gamma$  and  $Q = \sum_\gamma Q_\gamma \mathbf{X}^\gamma$  where  $\gamma$  runs over elements in  $\mathbf{N}_0^{n+1}$  with  $|\gamma|_1 = a$ , then

$$\langle P, Q \rangle = \sum_\gamma \binom{a}{\gamma}^{-1} P_\gamma \tau(Q_\gamma) \in K.$$

If  $b \in \mathbf{N}_0$ , then we may identify  $K[\mathbf{X}, \mathbf{Y}]_{(a,b)}$  with  $K[\mathbf{X}]_a \otimes_K K[\mathbf{Y}]_b$ . Thus we obtain a  $\tau$ -inner product on  $K[\mathbf{X}, \mathbf{Y}]_{(a,b)}$ .

In the next two lemmas,  $K$  is an algebraically closed field of characteristic 0,  $\tau$  is an involution on  $K$ , and  $\langle \cdot, \cdot \rangle$  is the  $\tau$ -inner product as given by Remark 5.2.

We recall that Segre and Veronese homomorphisms were defined in Subsection 4.1.

**Lemma 5.2.** *The Segre homomorphism  $s^* : K[\mathbf{U}]_k \rightarrow K[\mathbf{X}, \mathbf{Y}]_{(k,k)}$  is a surjective isometry for all  $k \in \mathbf{N}$ .*

*Proof.* We fix  $k \in \mathbf{N}$ . Surjectivity holds because elements in the target are bihomogeneous of bidegree  $(k, k)$ .

For brevity we set  $N = (n+1)(r+1)$ . Let  $P, Q \in K[\mathbf{U}]_k$  and  $P \in (\ker s^*)^\perp$ , we must show  $\langle s^*(P), s^*(Q) \rangle = \langle P, Q \rangle$ . We write  $P = \sum_\gamma P_\gamma \mathbf{U}^\gamma$  and  $Q = \sum_\gamma Q_\gamma \mathbf{U}^\gamma$ , where here and below the sum is over all  $\gamma \in \mathbf{N}_0^N$  with  $|\gamma|_1 = k$ .

We call  $\gamma, \gamma_0 \in \mathbf{N}_0^N$  with  $|\gamma|_1 = |\gamma_0|_1 = k$  equivalent, and write  $\gamma \sim \gamma_0$ , if and only if  $s^*(\mathbf{U}^\gamma) = s^*(\mathbf{U}^{\gamma_0})$ . Let  $R \subset \mathbf{N}_0^N$  be a set of representatives of the equivalence classes.

Say  $\gamma \sim \gamma_0$ . Then  $P \in (\ker s^*)^\perp$  implies  $\langle P, \mathbf{U}^\gamma \rangle = \langle P, \mathbf{U}^{\gamma_0} \rangle$ , so

$$(5.2) \quad \binom{k}{\gamma}^{-1} P_\gamma = \binom{k}{\gamma_0}^{-1} P_{\gamma_0}.$$

We have  $\langle s^*(P), s^*(Q) \rangle = \sum_{\gamma_0 \in R} \langle \sum_{\gamma \sim \gamma_0} P_\gamma s^*(\mathbf{U}^\gamma), \sum_{\gamma \sim \gamma_0} Q_\gamma s^*(\mathbf{U}^\gamma) \rangle$  because  $s^*(\mathbf{U}^\gamma)$  and  $s^*(\mathbf{U}^{\gamma_0})$  are orthogonal if  $\gamma \not\sim \gamma_0$ . Equality (5.2) gives

$$(5.3) \quad \begin{aligned} \langle s^*(P), s^*(Q) \rangle &= \sum_{\gamma_0 \in R} \left( \sum_{\gamma \sim \gamma_0} P_\gamma \right) \left( \sum_{\gamma \sim \gamma_0} \tau(Q_\gamma) \right) \langle s^*(\mathbf{U}^{\gamma_0}), s^*(\mathbf{U}^{\gamma_0}) \rangle \\ &= \sum_{\gamma_0 \in R} \binom{k}{\gamma_0}^{-1} P_{\gamma_0} \left( \sum_{\gamma \sim \gamma_0} \binom{k}{\gamma} \right) \left( \sum_{\gamma \sim \gamma_0} \tau(Q_\gamma) \right) \langle s^*(\mathbf{U}^{\gamma_0}), s^*(\mathbf{U}^{\gamma_0}) \rangle. \end{aligned}$$

Say  $s^*(\mathbf{U}^{\gamma_0}) = \mathbf{X}^\delta \mathbf{Y}^{\delta'}$  with  $\delta \in \mathbf{N}_0^{n+1}$ ,  $\delta' \in \mathbf{N}_0^{r+1}$ , and  $|\delta|_1 = |\delta'|_1 = k$ . By definition we have  $\langle s^*(\mathbf{U}^{\gamma_0}), s^*(\mathbf{U}^{\gamma_0}) \rangle = \langle \mathbf{X}^\delta, \mathbf{X}^\delta \rangle \langle \mathbf{Y}^{\delta'}, \mathbf{Y}^{\delta'} \rangle = \binom{k}{\delta}^{-1} \binom{k}{\delta'}^{-1}$ . On expanding both sides of  $s^*((\sum_{ij} U_{ij})^k) = (\sum_{ij} X_i Y_j)^k$  and comparing coefficients we find  $\left( \sum_{\gamma \sim \gamma_0} \binom{k}{\gamma} \right) \langle s^*(\mathbf{U}^{\gamma_0}), s^*(\mathbf{U}^{\gamma_0}) \rangle = 1$ . Therefore, (5.3) gives  $\langle s^*(P), s^*(Q) \rangle = \sum_{\gamma_0 \in R} \sum_{\gamma \sim \gamma_0} \binom{k}{\gamma_0}^{-1} P_{\gamma_0} \tau(Q_\gamma)$ . But the right-hand side is  $\langle P, Q \rangle$  because of (5.2).  $\square$

We can state an analog result for the Veronese homomorphism. The proof goes along similar lines as well.

**Lemma 5.3.** *The Veronese map  $v_a^* : K[\tilde{\mathbf{X}}]_k \rightarrow K[\mathbf{X}]_{ak}$  is a surjective isometry for all  $a, k \in \mathbf{N}$ .*

*Proof.* We fix  $a, k \in \mathbf{N}$ . Surjectivity follows immediately. Let  $P, Q \in K[\tilde{\mathbf{X}}]_k$  and  $P \in (\ker v_a^*)^\perp$ , we must show  $\langle v_a^*(P), v_a^*(Q) \rangle = \langle P, Q \rangle$ .

For brevity let  $N = \binom{n+a}{n}$ . We recall that  $\tilde{\mathbf{X}}$  is an  $N$ -tuple of independent variables  $(X_\gamma)$  where  $\gamma$  runs over elements of  $\mathbf{N}_0^{n+1}$  with  $|\gamma|_1 = a$ . We use  $\gamma$  to index elements of  $\mathbf{N}_0^N$ . If  $\delta \in \mathbf{N}_0^N$  with  $|\delta|_1 = k$  then

$$v_a^*(\tilde{\mathbf{X}}^\delta) = \mathbf{X}^{\alpha(\delta)} \prod_\gamma \binom{a}{\gamma}^{\delta_\gamma/2}$$

for some  $\alpha(\delta) \in \mathbf{N}_0^{n+1}$ , here and below  $\gamma$  runs over all elements of  $\mathbf{N}_0^{n+1}$  with  $|\gamma|_1 = a$ .

We call  $\delta, \delta_0 \in \mathbf{N}_0^N$  with  $|\delta|_1 = |\delta_0|_1 = k$  equivalent, and write  $\delta \sim \delta_0$ , if and only if  $\alpha(\delta) = \alpha(\delta_0)$ . I.e. if and only if  $v_a^*(\tilde{\mathbf{X}}^\delta)$  and  $v_a^*(\tilde{\mathbf{X}}^{\delta_0})$  are equal up-to a factor in  $K^\times$ . Let  $R \subset \mathbf{N}_0^N$  be a set of representatives of the equivalence classes.

Let  $P = \sum_\delta P_\delta \tilde{\mathbf{X}}^\delta$  and  $Q = \sum_\delta Q_\delta \tilde{\mathbf{X}}^\delta$  where here and below  $\delta$  runs over all elements of  $\mathbf{N}_0^N$  with  $|\delta|_1 = k$ .

Say  $\delta \sim \delta_0$ . Our assumption  $P \in (\ker v_a^*)^\perp$  implies

$$\left\langle P, \tilde{\mathbf{X}}^\delta \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{\gamma/2}} \right\rangle = \left\langle P, \tilde{\mathbf{X}}^{\delta_0} \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{0\gamma/2}} \right\rangle.$$

So

$$P_\delta \binom{k}{\delta}^{-1} \tau \left( \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{\gamma/2}} \right) = P_{\delta_0} \binom{k}{\delta_0}^{-1} \tau \left( \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{0\gamma/2}} \right).$$

and because  $\tau$  acts trivially on roots of positive integers we obtain

$$(5.4) \quad P_\delta \binom{k}{\delta}^{-1} \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{\gamma/2}} = P_{\delta_0} \binom{k}{\delta_0}^{-1} \prod_{\gamma} \binom{a}{\gamma}^{-\delta_{0\gamma/2}}.$$

We note that  $\langle v_a^*(\tilde{\mathbf{X}}^\delta), v_a^*(\tilde{\mathbf{X}}^{\delta_0}) \rangle = 0$  if  $\delta \not\sim \delta_0$  and evaluate

$$\begin{aligned} \langle v_a^*(P), v_a^*(Q) \rangle &= \sum_{\delta_0 \in R} \left\langle \sum_{\delta \sim \delta_0} P_\delta v_a^*(\tilde{\mathbf{X}}^\delta), \sum_{\delta \sim \delta_0} Q_\delta v_a^*(\tilde{\mathbf{X}}^\delta) \right\rangle \\ &= \sum_{\delta_0 \in R} \frac{P_{\delta_0}}{\binom{k}{\delta_0} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{0\gamma/2}}} \left\langle \mathbf{X}^{\alpha(\delta_0)} \sum_{\delta \sim \delta_0} \binom{k}{\delta} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma}} , \mathbf{X}^{\alpha(\delta_0)} \sum_{\delta \sim \delta_0} Q_\delta \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma/2}} \right\rangle \end{aligned}$$

after applying (5.4). The definition of the inner product shows that  $\langle v_a^*(P), v_a^*(Q) \rangle$  equals

$$\sum_{\delta_0 \in R} \frac{P_{\delta_0}}{\binom{k}{\delta_0} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{0\gamma/2}}} \binom{ak}{\alpha(\delta_0)}^{-1} \left( \sum_{\delta \sim \delta_0} \binom{k}{\delta} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma}} \right) \left( \sum_{\delta \sim \delta_0} \tau(Q_\delta) \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma/2}} \right).$$

We evaluate both sides of  $v_a^* \left( \left( \sum_{\gamma} \binom{a}{\gamma}^{1/2} \tilde{X}_\gamma \right)^k \right) = \left( \sum_{\gamma} \binom{a}{\gamma} \mathbf{X}^\gamma \right)^k = \left( \sum_{i=0}^n X_i \right)^{ak}$  and compare coefficients to deduce  $\sum_{\delta \sim \delta_0} \binom{k}{\delta} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma}} = \binom{ak}{\alpha(\delta_0)}$ . Hence

$$\langle v_a^*(P), v_a^*(Q) \rangle = \sum_{\delta_0 \in R} \frac{P_{\delta_0}}{\binom{k}{\delta_0} \prod_{\gamma} \binom{a}{\gamma}^{\delta_{0\gamma/2}}} \left( \sum_{\delta \sim \delta_0} \tau(Q_\delta) \prod_{\gamma} \binom{a}{\gamma}^{\delta_{\gamma/2}} \right) = \sum_{\delta} \frac{P_\delta \tau(Q_\delta)}{\binom{k}{\delta}}$$

where the second inequality follows from (5.4). But the right-hand side is just  $\langle P, Q \rangle$  by definition.  $\square$

Let  $a, b \in \mathbf{N}$  and let  $\tilde{\mathbf{Y}}$  be the  $\binom{r+b}{b}$ -tuple  $(\tilde{Y}_\gamma)$  where  $\gamma$  runs through all elements in  $\mathbf{N}_0^{r+1}$  with  $|\gamma|_1 = b$ . A consequence Lemmas 5.1 and 5.3 is that the linear map

$$v_{ab}^* = v_a^* \otimes v_b^* : K[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]_{(k,k)} \rightarrow K[\mathbf{X}, \mathbf{Y}]_{ak,bk}$$

is a surjective isometry; we identified  $K[\tilde{\mathbf{X}}]_k \otimes_K K[\tilde{\mathbf{Y}}]_k = K[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]_{(k,k)}$  and  $K[\mathbf{X}]_{ak} \otimes_K K[\mathbf{Y}]_{bk} = [\mathbf{X}, \mathbf{Y}]_{(ak,bk)}$ .

The next lemma is an application to an irreducible closed subvariety  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  defined over  $\overline{\mathbf{Q}}$ . We recall that  $s$  is the Segre morphism defined in Subsection 4.1.

**Lemma 5.4.** *We have*

$$\mathcal{H}_a(k, k; Z) \leq \mathcal{H}_a(k; s(Z)) \quad \text{and} \quad \mathcal{H}_g(k, k; Z) = \mathcal{H}_g(k; s(Z)).$$

for all  $k \in \mathbf{N}$ .



*Proof.* Let  $I \subset \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$  be the ideal of  $Z$  and  $J \subset \overline{\mathbf{Q}}[\mathbf{U}]$  the ideal of  $s(Z)$ . Then

$$(5.5) \quad s^{*-1}(I_{(k,k)}) = J_k.$$

Remark 5.2 gives us a  $\tau$ -inner product  $\langle \cdot, \cdot \rangle$  on  $\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(k,k)}$  and  $\overline{\mathbf{Q}}[\mathbf{U}]_k$  with  $\tau$  the identity on  $\overline{\mathbf{Q}}$ .

We now fix a basis  $\{Q_1, \dots, Q_t\}$  of  $J_k^\perp$  where  $t = \mathcal{H}_g(k; S(Z)) \geq 1$ . We define

$$A = [\iota(Q_1), \dots, \iota(Q_t)] \in \text{Mat}_{\binom{nr+n+r+k}{k}, t}(\overline{\mathbf{Q}}) \quad \text{and} \\ C = [\iota(s^*(Q_1)), \dots, \iota(s^*(Q_t))] \in \text{Mat}_{\binom{n+k}{k} \binom{r+k}{k}, t}(\overline{\mathbf{Q}}).$$

Let us remark that  $\ker s^* \subset J_k$  by (5.5). This implies  $(\ker s^*)^\perp \supset J_k^\perp$  and hence  $Q_i \in (\ker s^*)^\perp$ . We claim

$$(5.6) \quad s^*(J_k^\perp) = I_{(k,k)}^\perp.$$

Indeed, if  $P \in I_{(k,k)}$  then there is  $Q \in J_k$  with  $s^*(Q) = P$ . Therefore,  $\langle s^*(Q_i), P \rangle = \langle Q_i, Q \rangle = 0$  for all  $i$  because  $s^*$  is an isometry by Lemma 5.2. Hence  $s^*(Q_i) \in I_{(k,k)}^\perp$  because  $P$  was arbitrary. This shows that the left side of (5.6) is contained in the right side. Equality (5.5) implies  $\dim J_k = \dim \ker s^* + \dim I_{(k,k)}$ . Moreover,  $\dim \ker s^* = \dim \overline{\mathbf{Q}}[\mathbf{U}]_k - \dim \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(k,k)}$  since  $s^*$  is surjective. It follows that  $J_k^\perp$  and  $I_{(k,k)}^\perp$  have equal dimension. But  $\ker s^* \cap J_k^\perp \subset \ker s^* \cap (\ker s^*)^\perp = 0$  by the comment shortly after (5.1). So  $s^*|_{J_k^\perp}$  is injective and hence  $\dim s^*(J_k^\perp) = \dim I_{(k,k)}^\perp$ .

This settles our claim (5.6) and the equality involving the geometric Hilbert function in the assertion. We also remark that  $C$  and  $A$  have equal rank  $t$ .

Let  $K \subset \overline{\mathbf{Q}}$  be a finite normal extension of  $\mathbf{Q}$  containing all of the finitely many algebraic numbers which are involved in the proof below. Let  $v \in M_K$ .

Say  $v$  is infinite and let  $\sigma = \sigma_v : K \rightarrow \mathbf{C}$  be an associated embedding. Since  $K$  is normal over  $\mathbf{Q}$  there is an automorphism  $\eta$  of  $K$  with  $\sigma(\eta(x)) = \overline{\sigma(x)}$  for all  $x \in K$ . By the Cauchy-Binet formula we have

$$\exp 2h_{v,t}(A) = \det(\sigma(A)^\top \overline{\sigma(A)}) = \det[\sigma(\iota(Q_i))^\top \overline{\sigma(\iota(Q_j))}]_{1 \leq i,j \leq t} = \sigma(\det[\iota(Q_i)^\top \eta(\iota(Q_j))])_{ij}$$

All coefficients involved in  $\iota$  are totally real algebraic numbers; hence invariant under  $\eta$ . We obtain  $\exp 2h_{v,t}(A) = \sigma(\det[\langle Q_i, \eta(Q_j) \rangle]_{ij})$ . Together with the Cauchy-Binet formula we also get

$$\begin{aligned} \exp 2h_{v,t}(C) &= \det(\sigma(C)^\top \overline{\sigma(C)}) = \det[\sigma(\iota(s^*(Q_i)))^\top \overline{\sigma(\iota(s^*(Q_j))})]_{ij} \\ &= \sigma(\det[\iota(s^*(Q_i))^\top \eta(\iota(s^*(Q_j))])_{ij} = \sigma(\det[\langle s^*(Q_i), s^*(\eta(Q_j)) \rangle]_{ij}). \end{aligned}$$

But  $s^*$  is an isometry so  $\exp 2h_{v,t}(C) = \sigma(\det[\langle Q_i, \eta(Q_j) \rangle]_{ij})$ , hence

$$(5.7) \quad h_{v,t}(C) = h_{v,t}(A).$$

Now say  $v$  is finite. If  $B$  is a matrix with  $m$  rows and  $1 \leq i_1 < \dots < i_t \leq m$  are integers, we let  $B_{i_1 \dots i_t}$  denote the submatrix of  $B$  consisting of the rows  $i_1, \dots, i_t$ . We fix  $i_1, \dots, i_t$  with  $|\det C_{i_1 \dots i_t}|_v = \exp h_{v,t}(C)$ . For  $1 \leq i \leq t$  we may find polynomials  $\tilde{Q}_i \in K[\mathbf{U}]_k$  whose terms are also terms of  $Q_i$  with the following property. The rows  $i_1, \dots, i_t$  of  $\tilde{C} = [\iota(s^*(\tilde{Q}_1)), \dots, \iota(s^*(\tilde{Q}_t))]$  equal the corresponding rows of  $C$  and all other rows are 0. The Cauchy-Binet formula implies  $\det(C^\top \tilde{C}) = \det(C_{i_1 \dots i_t})^2$ .

On the other hand, we have  $\det(C^\top \tilde{C}) = \det[\langle s^* Q_i, s^* \tilde{Q}_j \rangle]_{ij} = \det[\langle Q_i, \tilde{Q}_j \rangle]_{ij}$  because  $s^*$  is an isometry. So  $\det(C^\top \tilde{C}) = \det(A^\top \tilde{A})$  with  $\tilde{A} = [\iota(\tilde{Q}_1), \dots, \iota(\tilde{Q}_t)]$ . Now we apply the Cauchy-Binet formula to evaluate  $\det(A^\top \tilde{A}) = \sum_{i'_1 < \dots < i'_t} \det(A_{i'_1 \dots i'_t}) \det(\tilde{A}_{i'_1 \dots i'_t})$ . Since a row of  $\tilde{A}$  either equals the corresponding row of  $A$  or is 0 we see that  $\det(\tilde{A}_{i'_1 \dots i'_t})$  is either  $\det(A_{i'_1 \dots i'_t})$  or 0. The ultrametric triangle inequality implies  $|\det C_{i_1 \dots i_t}|_v^2 = |\det C^\top \tilde{C}|_v = |\det(A^\top \tilde{A})|_v \leq \max_{i'_1 < \dots < i'_t} |\det A_{i'_1 \dots i'_t}|_v^2 = \exp(2h_{v,t}(A))$ . We conclude

$$(5.8) \quad h_{v,t}(C) \leq h_{v,t}(A).$$

We multiply (5.7) and (5.8) with  $[K_v : \mathbf{Q}_v]/[K : \mathbf{Q}]$  and take the sum over all places of  $K$  to obtain

$$(5.9) \quad h_t(C) \leq h_t(A).$$

By definition we have  $\mathcal{H}_a(k, k; Z) = h_{\text{Ar}}(I_{(k,k)})$ . We know how the height behaves under taken the orthogonal complement, see (3.5) in Remark 3.2. We conclude  $\mathcal{H}_a(k, k; Z) = h_{\text{Ar}}(I_{(k,k)}^\perp)$ . The columns of  $C$  are a basis of  $\iota(I_{(k,k)}^\perp)$  by (5.6), so  $\mathcal{H}_a(k, k; Z) = h_t(C)$ . On the other hand,  $\mathcal{H}_a(k, s(Z)) = h_{\text{Ar}}(J_k) = h_{\text{Ar}}(J_k^\perp) = h_t(A)$  by similar arguments. The proof follows from (5.9).  $\square$

**Lemma 5.5.** *We have*

$$\mathcal{H}_a(ak, bk; Z) \leq \mathcal{H}_a(k, k; v_{ab}(Z)) \quad \text{and} \quad \mathcal{H}_g(ak, bk; Z) = \mathcal{H}_g(k, k; v_{ab}(Z))$$

for  $a, b, k \in \mathbf{N}$ .

*Proof.* Let  $I$  be the ideal of  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  and let  $J$  be the ideal of the closed subvariety  $v_{ab}(Z) \subset \mathbf{P}^{\binom{n+a}{a}-1} \times \mathbf{P}^{\binom{r+b}{b}-1}$ . Following the convention that  $v_{ab}^*$  denotes the restriction of the Veronese homomorphism to polynomials of fixed bidegree, we have

$$v_{ab}^*{}^{-1}(I_{(ak,bk)}) = J_{(k,k)}.$$

We mainly follow the lines of the proof of Lemma 5.4. Remark 5.2 gives us a  $\tau$ -inner product  $\langle \cdot, \cdot \rangle$  on source and target of  $v_{ab}^*$  with  $\tau$  the identity on  $\overline{\mathbf{Q}}$ . Let us fix a basis  $\{Q_1, \dots, Q_t\}$  of  $J_{(k,k)}^\perp$  with  $t = \mathcal{H}_g(k, k; v_{ab}(Z))$ . We also introduce matrices

$$A = [\iota(Q_1), \dots, \iota(Q_t)] \in \text{Mat}_{\binom{n+a}{a}-1+k, \binom{r+b}{b}-1+k, t}(\overline{\mathbf{Q}}) \quad \text{and} \\ C = [\iota(v_{ab}^*(Q_1)), \dots, \iota(v_{ab}^*(Q_t))] \in \text{Mat}_{\binom{n+ak}{ak}, \binom{r+bk}{bk}, t}(\overline{\mathbf{Q}}).$$

We have  $J_{(k,k)} \supset \ker v_{ab}^*$ , so  $J_{(k,k)}^\perp \subset (\ker v_{ab}^*)^\perp$  and hence  $Q_i \in (\ker v_{ab}^*)^\perp$ .

As in the proof of Lemma 5.4 we use the fact that  $v_{ab}^*$  is an isometry, cf. Lemma 5.3, to deduce

$$v_{ab}^*(J_{(k,k)}^\perp) = I_{(ak,bk)}^\perp$$

as well as the second claim of the current lemma and also that  $C$  has rank  $t$ .

Again we let  $K$  be a finite normal extension of  $\mathbf{Q}$  which contains all algebraic numbers which follow. Let  $v \in M_K$ .

Say  $v$  is infinite. Just as in Lemma 5.4 we have

$$h_{v,t}(C) = h_{v,t}(A);$$

in order to show this we need to use the fact that  $v_{ab}^*$  is an isometry but also that its coefficients are totally real numbers. Now say  $v$  is finite. Using a similar argument as in Lemma 5.4 we find

$$h_{v,t}(C) \leq h_{v,t}(A).$$

The remainder of the proof is just as in Lemma 5.4.  $\square$

## 6. CORRESPONDENCES AND HEIGHT INEQUALITIES

The goal of this section is to prove a height inequality on correspondences.

Let  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  be an irreducible closed subvariety of positive dimension defined over  $\overline{\mathbf{Q}}$ . Recall that  $s : \mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^{nr+n+r}$  is the Segre morphism. We define the height  $h(Z)$  and degree  $\deg(Z)$  of  $Z$  as height and degree of  $s(Z)$ . Recall that  $\pi_1$  and  $\pi_2$  are the two projections  $\mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^n$  and  $\mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^r$ . We may attach to  $Z$  a  $(1 + \dim Z)$ -tuple of bidegrees

$$\Delta_i(Z) = (\pi_1^* \mathcal{O}(1)^i \pi_2^* \mathcal{O}(1)^{\dim Z - i} [Z]) \geq 0 \quad \text{for } i \in \{0, \dots, \dim Z\}.$$

In this notation, the Hilbert polynomial of  $Z$  is

$$(6.1) \quad H(T_1, T_2; Z) = \sum_{i=0}^{\dim Z} \binom{\dim Z}{i} \Delta_i(Z) T_1^i T_2^{\dim Z - i}.$$

We introduce two restrictions on  $Z$ .

- (i) We suppose  $\Delta_0(Z) > 0$ . This is equivalent to stating that  $\pi_2|_Z : Z \rightarrow \mathbf{P}^r$  is generically finite.
- (ii) We suppose there exists  $i \in \{1, \dots, \dim Z\}$  with  $\Delta_i(Z) > 0$ .

Property (i) implies  $\dim Z \leq r$ . Property (ii) excludes examples such as  $Z = \{p\} \times Z'$  for some irreducible closed subvariety  $Z' \subset \mathbf{P}^r$ .

If  $Z$  satisfies conditions (i) and (ii) above, we define

$$(6.2) \quad \kappa(Z) = \min \left\{ \frac{\Delta_0(Z)}{\Delta_1(Z)}, \left( \frac{\Delta_0(Z)}{\Delta_2(Z)} \right)^{1/2}, \dots, \left( \frac{\Delta_0(Z)}{\Delta_d(Z)} \right)^{1/d} \right\}.$$

Note that certain quotients can be infinity. But the minimum is always a positive real number by (i) and (ii).

**Proposition 6.1.** *Let  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $d \geq 1$ . We assume that  $Z$  satisfies conditions (i) and (ii) above. Let  $\kappa = \kappa(Z)$  and  $\Delta_0 = \Delta_0(Z)$ . For brevity, we introduce the constant*

$$k_0 = \max\{17 \cdot 3^d d! n \Delta_0^{d-1}, \deg(Z)\} \times \begin{cases} 1 & : \text{if } \kappa \geq 17dn \text{ and} \\ 100 \frac{dn}{\kappa} & : \text{elsewise.} \end{cases}$$

There exist  $a, b \in \mathbf{N}$  with  $\max\{a, b\} \leq k_0$  and

$$a = b \quad \text{if } \kappa \geq 17dn$$

and  $F \in \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,b)}$  with

$$h(F) \leq \left( 400n \max\{n, r\}^2 \frac{\max\{1, \kappa\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 5d \deg(Z) \right) k_0$$

such that the following properties hold. The polynomial  $F$  does not vanish identically on  $Z$  and if  $(p, q) \in Z(\overline{\mathbf{Q}})$  then we are in one of the following cases.

- (i) We have  $F(p, q) = 0$  or some projective coordinate of  $\mathbf{P}^n$  vanishes at  $p$ .
- (ii) We have the inequality

$$\kappa h(p) \leq 2^5 d n h(q) + 2^{14} n^2 r \max\{n, r\}^2 \frac{\max\{1, \kappa\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 2^8 n r^2 \deg(Z).$$

For our application a slightly modified version of the previous proposition will be of central importance.

**Proposition 6.2.** *Let  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $d \geq 1$ . We assume that  $Z$  satisfies conditions (i) and (ii) above. Let  $\kappa = \kappa(Z)$  and  $\Delta_0 = \Delta_0(Z)$ , we shall assume  $\kappa \geq 17dn$ . We set  $k_0 = \max\{17 \cdot 3^d d! n \Delta_0^{d-1}, \deg(Z)\}$ . There is a finite collection of irreducible closed subvarieties  $V_1, \dots, V_N \subset \pi_1(X) \subset \mathbf{P}^n$  defined over  $\overline{\mathbf{Q}}$  with*

$$\dim V_i = d - 1,$$

$$(6.3) \quad \sum_{i=1}^N \deg(V_i) \leq k_0 \deg(Z), \quad \text{and}$$

$$h(V_i) \leq 2^9 \max\{n^2 r^2, n r^3, n^3 r\} \max\left\{1, \frac{\kappa^{d+1}}{\Delta_0}\right\} (h(Z) + \deg(Z)) \deg(Z) k_0,$$

and such that the following property holds. Let  $(p, q) \in Z(\overline{\mathbf{Q}})$ , then we are in one of the following cases.

- (i) Some projective coordinate of  $\mathbf{P}^n$  vanishes at  $p$  or  $(p, q)$  is not isolated in  $\pi_1|_Z^{-1}(p)$ .
- (ii) There is  $1 \leq i \leq N$  with  $p \in V_i(\overline{\mathbf{Q}})$ .
- (iii) We have

$$\kappa h(p) \leq 2^5 d n h(q) + 2^{15} \max\{n^4 r, n^2 r^3\} \max\left\{1, \frac{\kappa^{d+1}}{\Delta_0}\right\} (h(Z) + \deg(Z)).$$

**6.1. Height Lower Bounds.** Let  $a, b \in \mathbf{N}_0$ . Recall that  $\mathbf{X} = (X_0, \dots, X_n)$  and  $\mathbf{Y} = (Y_0, \dots, Y_r)$  and that we defined a height of  $(n+1)$ -tuples  $F = (F_0, \dots, F_n) \in (\overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,b)})^{n+1}$  in Section 3.2.

Until the end of Section 6 we will use the following notation. We let  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $d \geq 1$ . We suppose that  $Z$  satisfies conditions (i) and (ii) introduced at the beginning of Section 6. We suppose additionally that the product  $X_0 \cdots X_n$  does not vanish identically on  $Z$ . Finally, for brevity we set  $\Delta_i = \Delta_i(Z)$  for  $0 \leq i \leq d$ .

The main tool for the height inequality below is the product formula.

**Lemma 6.1.** *Suppose  $I \subset \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$  is the ideal of  $Z$ . Let  $F = (F_0, \dots, F_n)$  be an  $(n+1)$ -tuple as above such that each  $F_i$  is bihomogeneous of bidegree  $(a, b)$  with  $a, b \in \mathbf{N}$ . We assume that there is a positive integer  $c$  such that*

$$X_i^c F_0 - X_0^c F_i \in I \quad \text{for } 1 \leq i \leq n.$$

Let  $(p, q) \in Z(\overline{\mathbf{Q}})$  where  $p = [p_0 : \cdots : p_n]$  and assume that  $p_0 \neq 0$  and  $F_0(p, q) \neq 0$ . Then

$$(c - a)h(p) \leq bh(q) + h(F) + \frac{c}{2} \log(n + 1).$$

*Proof.* Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing algebraic projective coordinates  $p_0, \dots, p_n$  and  $q_0, \dots, q_r$  of  $p$  and  $q$ , respectively. We also assume that  $K$  contains all of the finitely many algebraic numbers implicit in the subsequent proof.

We write  $x = (p_0, \dots, p_n) \in K^{n+1}$ ,  $y = (q_0, \dots, q_r) \in K^{r+1}$ , and  $z = (F_0(x, y), \dots, F_n(x, y)) \in K^{n+1} \setminus \{0\}$ .

For  $v \in M_K$  we set

$$\lambda(v) = \log \frac{|x|_v^a |y|_v^b}{|z|_v}.$$

The definition of  $|F_i|_v$  was given in Section 3.2. The Cauchy-Schwarz inequality gives  $|F_i(x, y)|_v \leq |F_i|_v |x|_v^a |y|_v^b$  if  $v$  is an infinite place of  $K$ . The same inequality holds for finite  $v$  by the ultrametric triangle inequality. For infinite  $v$  we have  $|z|_v = (|F_0(x, y)|_v^2 + \cdots + |F_n(x, y)|_v^2)^{1/2} \leq (|F_0|_v^2 + \cdots + |F_n|_v^2)^{1/2} |x|_v^a |y|_v^b = |F|_v |x|_v^a |y|_v^b$ . And for finite  $v$  the corresponding statement is  $|z|_v = \max\{|F_0(x, y)|_v, \dots, |F_n(x, y)|_v\} \leq |F|_v |x|_v^a |y|_v^b$ .

These bounds imply

$$(6.4) \quad \lambda(v) \geq -\log |F|_v$$

regardless if  $v$  is finite or not.

The hypothesis implies  $x_i^c F_0(x, y) = x_0^c F_i(x, y)$  for  $0 \leq i \leq n$ , therefore

$$z = F_0(x, y) x_0^{-c} (x_0^c, \dots, x_n^c).$$

Hence

$$\lambda(v) = \log \left| \frac{x_0^c}{F_0(x, y)} \right|_v + \log \frac{|x|_v^a |y|_v^b}{|(x_0^c, \dots, x_n^c)|_v}.$$

If  $v$  is finite, then  $|(x_0^c, \dots, x_n^c)|_v = |x|_v^c$  and if  $v$  is infinite we may estimate  $|x|_v^{2c} = (\sum_{i=0}^n |x_i|_v^2)^c \leq (n+1)^c |(x_0^c, \dots, x_n^c)|_v^2$ .

Now we sum over all places of  $K$  weighted with the appropriate local degree to obtain

$$\begin{aligned} \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \lambda(v) &\leq \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log \left| \frac{x_0^c}{F_0(x, y)} \right|_v \\ &\quad + \sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \log |x|_v^{a-c} |y|_v^b + \sum_{v \text{ infinite}} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} \frac{c}{2} \log(n+1). \end{aligned}$$

The first term on the right side of the inequality is zero by the product formula. The second term is  $(a - c)h(p) + bh(q)$  by the definition of our height. The final term is  $c/2 \log(n+1)$  since  $\sum_{v \text{ infinite}} [K_v : \mathbf{Q}_v] = [K : \mathbf{Q}]$ . We use (6.4) to bound the left side of the inequality from below. This completes the proof.  $\square$

**6.2. Dimension Inequalities.** For brevity we set  $R = \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]$ . Let  $I \subset R$  be the ideal of  $Z \subset \mathbf{P}^n \times \mathbf{P}^r$ .

By our convention, stated in Section 3.1,  $\overline{\mathbf{Q}}$  is a subfield of  $\mathbf{C}$ . We now take  $\tau$  to be complex conjugation restricted to  $\overline{\mathbf{Q}}$ . It acts trivially on square roots of positive integers. So it satisfies the restrictions imposed in Remark 5.2

In the following sections,  $\langle \cdot, \cdot \rangle$  is the  $\tau$ -inner product as in said remark on vector spaces of polynomials with coefficients in  $\overline{\mathbb{Q}}$ .

*Remark 6.1.* Suppose  $F \in I_{(a,b)}$ . Since  $\tau$  is the restriction of complex conjugation, the inner product  $\langle F, F \rangle$  vanishes if and only if  $F = 0$ . Therefore, we have  $I_{(a,b)} \cap I_{(a,b)}^\perp = 0$ .

Until the end of this section we treat  $D, E, k \in \mathbb{N}$  as parameters. They will be chosen later on. We set

$$(6.5) \quad W_{DEk} = \{(F_0, \dots, F_n) \in (I_{(k,Ek)}^\perp)^{n+1}; X_i^{Dk} F_0 - X_0^{Dk} F_i \in I \text{ for } 1 \leq i \leq n\}.$$

Any  $F = (F_0, \dots, F_n) \in W_{DEk}$  satisfies the hypothesis of Lemma 6.1 with  $(a, b) = (k, Ek)$  and  $c = Dk$ .

Our general strategy is a classical step in many proofs in diophantine approximation. We shall find a non-zero element of small height in  $W_{DEk}$ . We carry out this strategy by applying an absolute version of Siegel's Lemma. Said element will then be used to apply Lemma 6.1.

Of course, we can only find non-zero elements in  $W_{DEk}$  if this vector space is non-trivial. Our first task will be to bound the dimension of  $W_{DEk}$  from below.

**Lemma 6.2.** *We have  $\dim W_{DEk} \geq (n+1)\mathcal{H}_g(k, Ek; Z) - n\mathcal{H}_g((D+1)k, Ek; Z)$ .*

*Proof.* The vector space  $W_{DEk}$  is the kernel of

$$\begin{aligned} (I_{(k,Ek)}^\perp)^{n+1} &\rightarrow (R_{((D+1)k, Ek)} / I_{((D+1)k, Ek)})^n \\ (F_0, \dots, F_n) &\mapsto (X_i^{Dk} F_0 - X_0^{Dk} F_i)_{1 \leq i \leq n}. \end{aligned}$$

Hence we have the dimension inequality

$$\begin{aligned} \dim W_{DEk} &\geq (n+1) \dim I_{(k,Ek)}^\perp - n \dim I_{((D+1)k, Ek)}^\perp \\ &= (n+1)\mathcal{H}_g(k, Ek; Z) - n\mathcal{H}_g((D+1)k, Ek; Z). \quad \square \end{aligned}$$

We will use the bounds developed in Section 4.1 to estimate the dimension of  $W_{DEk}$  in terms of the bidegree  $\Delta_0$ .

**Lemma 6.3.** *We assume  $k \geq \Delta_0$  and*

$$\frac{D+1}{E} \leq \frac{1}{4dn} \kappa(Z).$$

*Then*

$$(6.6) \quad H(D+1, E; Z) \leq \left(1 + \frac{1}{2n}\right) \Delta_0 E^d$$

*and*

$$\frac{1}{\Delta_0 E^d} \dim W_{DEk} \geq \frac{1}{2} \frac{k^d}{d!} - 4ne^d \Delta_0^{d-1} k^{d-1}.$$

*Proof.* We begin by showing (6.6). We use (6.1) to estimate values of the Hilbert polynomial. It suffices to show the inequality in

$$(6.7) \quad \frac{1}{E^d} H(D+1, E; Z) - \Delta_0 = \sum_{i=1}^d \binom{d}{i} \Delta_i \left(\frac{D+1}{E}\right)^i \stackrel{?}{\leq} \frac{\Delta_0}{2n}.$$



We set  $\delta_i = (\Delta_0/\Delta_i)^{1/i}$  if  $\Delta_i \neq 0$ ; by hypothesis we have  $(D+1)/E \leq \delta_i/(4dn)$ . We deduce

$$\sum_{i=1}^d \binom{d}{i} \Delta_i \left( \frac{D+1}{E} \right)^i \leq \sum_{\substack{i=1 \\ \Delta_i \neq 0}}^d \binom{d}{i} \Delta_i \delta_i^i \frac{1}{(4dn)^i} \leq \Delta_0 \sum_{i=1}^d \binom{d}{i} \frac{1}{(4dn)^i}.$$

It suffices to show that the right-hand side is at most  $\Delta_0/(2n)$ . We have

$$\sum_{i=1}^d \binom{d}{i} \frac{1}{(4dn)^i} = \left( 1 + \frac{1}{4dn} \right)^d - 1 \leq \exp \left( \frac{1}{4n} \right) - 1$$

since  $(1 + \frac{1}{4dn})^d$  is increasing in  $d$  with limit  $\exp(1/(4n))$ . Elementary estimates show  $e^{1/(4n)} - 1 \leq 1/(2n)$  and our claim (6.7) follows.

We come to the second part of the lemma. By Lemma 4.5 we have

$$\mathcal{H}_g(k, Ek; Z) \geq \Delta_0 \binom{d + Ek - \Delta_0}{d} \geq \frac{\Delta_0}{d!} (Ek - \Delta_0)^d.$$

Expanding the expression on the right and isolating the term  $(Ek)^d$  gives

$$\begin{aligned} (6.8) \quad \mathcal{H}_g(k, Ek; Z) &\geq \Delta_0 \frac{(Ek)^d}{d!} + \sum_{i=0}^{d-1} \binom{d}{i} (Ek)^i (-\Delta_0)^{d-i} \\ &\geq \Delta_0 \frac{(Ek)^d}{d!} - (Ek)^{d-1} \sum_{i=0}^{d-1} \binom{d}{i} \Delta_0^{d-i} \geq \Delta_0 \frac{(Ek)^d}{d!} - (Ek)^{d-1} (2\Delta_0)^d. \end{aligned}$$

On the other hand, Lemma 4.4 implies the upper bound

$$\mathcal{H}_g((D+1)k, Ek; Z) \leq \binom{d+k}{d} H(D+1, E; Z) \leq \left( \frac{k^d}{d!} + e^d k^{d-1} \right) H(D+1, E; Z);$$

the second inequality follows from basic calculus where  $e = 2.71828\dots$ . If we apply inequality (6.8) to the conclusion of Lemma 6.2 we get

$$\begin{aligned} \dim W_{DEk} &\geq \frac{1}{2} \mathcal{H}_g(k, Ek; Z) + n \left( \left( 1 + \frac{1}{2n} \right) \mathcal{H}_g(k, Ek; Z) - \mathcal{H}_g((D+1)k, Ek; Z) \right) \\ &\geq \frac{\Delta_0}{2} \frac{(Ek)^d}{d!} + n \frac{k^d}{d!} \left( \left( 1 + \frac{1}{2n} \right) \Delta_0 E^d - H(D+1, E; Z) \right) + \\ &\quad - (n+1)(Ek)^{d-1} (2\Delta_0)^d - ne^d k^{d-1} H(D+1, E; Z). \end{aligned}$$

Because of the first statement of this lemma, the second term on the very right of the inequality is non-negative. This statement also controls the remaining  $H(D+1, E; Z)$ , hence

$$\begin{aligned} \dim W_{DEk} &\geq \frac{\Delta_0}{2} \frac{(Ek)^d}{d!} - (n+1)(Ek)^{d-1} (2\Delta_0)^d - ne^d k^{d-1} H(D+1, E; Z) \\ &\geq \frac{\Delta_0}{2} \frac{(Ek)^d}{d!} - (n+1)(Ek)^{d-1} (2\Delta_0)^d - \left( n + \frac{1}{2} \right) e^d k^{d-1} \Delta_0 E^d. \end{aligned}$$

The proof follows from  $(n+1)(Ek)^{d-1} (2\Delta_0)^d + (n+1/2)e^d k^{d-1} \Delta_0 E^d \leq 4ne^d \Delta_0^d E^d k^{d-1}$ .  $\square$

In particular, if  $D$  and  $E$  satisfy the lemma's hypothesis, then  $W_{DEk} \neq 0$  for large  $k$ .

**6.3. Bounding the Height of  $W_{DEk}$ .** In order to apply Siegel's Lemma to  $W_{DEk}$  we must also bound its height from above. It turns out to be easier to work with a vector space  $W'_{DEk}$  closely related to  $W_{DEk}$  which we proceed to define.

For each  $i \in \{0, \dots, n\}$  we set

$$V_i = \{X_i^{Dk} F; F \in I_{(k, Ek)}^\perp\} \subset R_{((D+1)k, Ek)}$$

and note that  $\dim V_i = \dim I_{(k, Ek)}^\perp = \mathcal{H}_g(k, Ek; Z)$ . We also define

$$W'_{DEk} = \{(G_1, G'_1, \dots, G_n, G'_n) \in \prod_{i=1}^n V_0 \times V_i; G_i - G'_i \in I \text{ for } 1 \leq i \leq n, \\ \frac{G'_1}{X_1^{Dk}} = \dots = \frac{G'_n}{X_n^{Dk}}\}.$$

We recall that the product  $X_0 \cdots X_n$  does not vanish identically on  $Z$ .

There is a homomorphism  $\Psi : W'_{DEk} \rightarrow W_{DEk}$  defined by

$$(6.9) \quad (G_1, G'_1, \dots, G_n, G'_n) \mapsto \left( \frac{G'_1}{X_1^{Dk}}, \frac{G_1}{X_0^{Dk}}, \frac{G_2}{X_0^{Dk}}, \dots, \frac{G_n}{X_0^{Dk}} \right).$$

It is readily checked to be injective. Moreover it has a right-inverse given by

$$(F_0, \dots, F_n) \mapsto (F_1 X_0^{Dk}, F_0 X_1^{Dk}, F_2 X_0^{Dk}, F_0 X_2^{Dk}, \dots, F_n X_0^{Dk}, F_0 X_n^{Dk}).$$

So  $\Psi$  is an isomorphism and  $\dim W'_{DEk} = \dim W_{DEk}$ .

We proceed by bounding the height of  $W'_{DEk}$ . To do this, we write our vector space as an intersection

$$W'_{DEk} = W_1 \cap W_2 \cap W_3$$

and bound the height of each  $W_i$  separately. The desired height bound will then follow from (3.4).

Explicitly, we set

$$W_1 = \prod_{i=1}^n V_0 \times V_i, \\ W_2 = \{(G_1, G'_1, \dots, G_n, G'_n) \in R_{((D+1)k, Ek)}^{2n}; G_i - G'_i \in I \text{ for } 1 \leq i \leq n\}, \quad \text{and} \\ W_3 = \{(G_1, G'_1, \dots, G_n, G'_n) \in R_{((D+1)k, Ek)}^{2n}; X_1^{Dk} G'_i = X_i^{Dk} G'_1 \text{ for } 1 \leq i \leq n\}.$$

A preliminary step in bounding the heights of the  $W_j$  is to relate the height of  $I_{(a,b)}^\perp$  with the value of the arithmetic Hilbert function at  $(a, b)$ . Later is just the height of  $I_{(a,b)}$ . The connection is a simple as one could hope for.

**Lemma 6.4.** *We have  $h_{\text{Ar}}(I_{(a,b)}^\perp) = \mathcal{H}_a(a, b; Z)$  for all  $a, b \in \mathbb{N}$ .*

*Proof.* We fix a basis  $\{Q_1, \dots, Q_t\}$  of  $I_{(a,b)}$ . A polynomial  $P$  lies in  $I_{(a,b)}^\perp$  if and only if  $\langle P, Q_i \rangle = \iota(P)^\top \cdot \tau(\iota(Q_i)) = 0$  for all  $1 \leq i \leq t$ . So  $\iota(I_{(a,b)}^\perp)$  is the kernel of the matrix  $A$  with  $s$  columns given by  $\tau(\iota(Q_i))$ . By (3.5) the height  $h_t(A)$  is  $h_{\text{Ar}}(\iota(I_{(a,b)}^\perp)) = h_{\text{Ar}}(I_{(a,b)}^\perp)$ . On the other hand,  $h_t(A) = h_t(\tau(A))$ . The columns of  $\tau(A)$  come from a basis of  $I_{(a,b)}$

since  $\tau(\tau(\iota(Q_i))) = \iota(Q_i)$ . Hence  $h_t(\tau(A)) = h_{\text{Ar}}(I_{(a,b)}) = \mathcal{H}_a(a, b; Z)$  and the proof is complete.  $\square$

**Lemma 6.5.** *We have*

$$h_{\text{Ar}}(W_1) \leq 2n\mathcal{H}_a(k, Ek; Z) + 20n \max\{n, r\}^2 \max\{D, E\} \mathcal{H}_g(k, Ek; Z)k.$$

*Proof.* Since  $W_1 = \prod_{i=1}^n V_0 \times V_i$  we use (3.3) to deduce

$$(6.10) \quad h_{\text{Ar}}(W_1) \leq nh_{\text{Ar}}(V_0) + \sum_{i=1}^n h_{\text{Ar}}(V_i) \leq 2n \max\{h_{\text{Ar}}(V_0), \dots, h_{\text{Ar}}(V_n)\}.$$

We continue by bounding the height of each  $V_i$ .

Let  $t = \dim(I_{(k, Ek)}^\perp) = \mathcal{H}_g(k, Ek; Z) \geq 1$  and let  $P_1, \dots, P_t$  be a basis of  $I_{(k, Ek)}^\perp$ . We define  $A$  to be the matrix whose  $t$  columns are  $\iota(P_1), \dots, \iota(P_t)$ .

A basis of  $V_i$  is given by  $X_i^{Dk}P_1, \dots, X_i^{Dk}P_t$ . Hence we may realize a basis for  $\iota(V_i)$  as the columns of the product  $BA$  where

$$B \in \text{Mat}_{\binom{n+(D+1)k}{n} \binom{r+Ek}{r}, \binom{n+k}{n} \binom{r+Ek}{r}}(\overline{\mathbf{Q}})$$

is a transformation matrix. The image of a monomial under the homomorphism  $P \mapsto X_i^{Dk}P$  is also a monomial. So each row of  $B$  has at most one non-zero entry. This entry is of the form

$$(6.11) \quad \binom{k}{\alpha}^{1/2} \binom{(D+1)k}{\alpha'}^{-1/2}$$

here  $\alpha, \alpha' \in \mathbf{N}_0^{n+1}$  correspond to monomials; they satisfy  $|\alpha|_1 = k$  and  $|\alpha'|_1 = (D+1)k$ .

Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing the finitely many algebraic numbers which appear in this proof and let  $v \in M_K$ .

Say  $v$  is infinite. The absolute value of (6.11) with respect to  $v$  is at most  $(n+1)^{k/2}$ . If  $B'$  is a  $t \times t$  submatrix of  $B$ , then  $|\det B'|_v \leq (n+1)^{tk/2}$ . The number of possibilities for  $B'$  is

$$\begin{aligned} \binom{\binom{n+(D+1)k}{n} \binom{r+Ek}{r}}{t} \binom{\binom{n+k}{n} \binom{r+Ek}{r}}{t} &\leq \binom{n+(D+1)k}{n}^t \binom{n+k}{n}^t \binom{r+Ek}{r}^{2t} \\ &\leq (n+(D+1)k)^{nt} (n+k)^{nt} (r+Ek)^{2rt}. \end{aligned}$$

The triangle inequality implies

$$h_{v,t}(B) \leq \frac{tk}{2} \log(n+1) + \frac{nt}{2} \log((n+(D+1)k)(n+k)) + rt \log(r+Ek).$$

Say  $v$  is finite place and let  $p$  be the rational prime with  $|p|_v < 1$ . Statement (3.1) gives

$$\left| \binom{k}{\alpha}^{1/2} \binom{(D+1)k}{\alpha'}^{-1/2} \right|_v \leq \begin{cases} ((D+1)k)^{(n+1)/2} & : \text{if } p \leq (D+1)k, \\ 1 & : \text{else wise.} \end{cases}$$

From the description of the entries of  $B$  given around (6.11) we deduce that

$$h_{v,t}(B) \leq \begin{cases} t \frac{n+1}{2} \log((D+1)k) & : \text{if } p \leq (D+1)k, \\ 1 & : \text{else wise.} \end{cases}$$

Now for arbitrary  $v$  we have  $h_{v,t}(BA) \leq h_{v,t}(B) + h_{v,t}(A)$  by Lemma 3.1. We multiply the local heights with the corresponding local degrees and sum over all places of  $K$  to obtain

$$h_t(BA) \leq \frac{tk}{2}(n+1) + \frac{nt}{2} \log((n+(D+1)k)(n+k)) + rt \log(r+Ek) \\ + t \frac{n+1}{2} \pi((D+1)k) \log((D+1)k) + h_t(A),$$

where  $\pi((D+1)k)$  denotes the number of rational primes at most  $(D+1)k$ . It is known that  $\pi((D+1)k) \log((D+1)k) \leq 2(D+1)k$ , see for example the work of Rosser and Schönfeld [41]. So

$$h_t(BA) \leq \frac{tk}{2}(n+1) + \frac{nt}{2} \log((n+(D+1)k)(n+k)) + rt \log(r+Ek) + (n+1)(D+1)tk + h_t(A) \\ \leq ntk + \frac{nt}{2} \log(6n^2Dk^2) + rt \log(2rEk) + 4nDtk + h_t(A) \\ \leq nt \log(\sqrt{6nDk}) + rt \log(2rEk) + 5nDtk + h_t(A) \\ \leq nt \log(\sqrt{6nD}) + ntk + rt \log(2rE) + rtk + 5nDtk + h_t(A).$$

We recall (6.10), together with  $h_t(BA) = h_{\text{Ar}}(V_i)$  this implies

$$h_{\text{Ar}}(W_1) \leq 2n^2t \log(\sqrt{6nD}) + 2n^2tk + 2nrt \log(2rE) + 2nrtk + 10n^2Dtk + 2nh_t(A).$$

Since  $h_t(A) = h_{\text{Ar}}(I_{(k,Ek)}^\perp) = \mathcal{H}_a(k, Ek; Z)$  by Lemma 6.4 we have

$$h_{\text{Ar}}(W_1) \leq 2n\mathcal{H}_a(k, Ek; Z) + (2n \log(\sqrt{6nD}) + 2n + 2r \log(2rE) + 2r + 10nD)ntk.$$

The lemma follows since  $t = \mathcal{H}_g(k, Ek; Z)$  and

$$2n \log(\sqrt{6nD}) + 2n + 2r \log(2rE) + 2r + 10nD \leq 2n^2D + 2n + 2r^2E + 2r + 10nD \\ \leq (2n^2 + 2n + 2r^2 + 2r + 10n) \max\{D, E\} \leq 18 \max\{n, r\}^2 \max\{D, E\}. \quad \square$$

**Lemma 6.6.** *We have*

$$h_{\text{Ar}}(W_2) \leq n\mathcal{H}_a((D+1)k, Ek; Z) + n\mathcal{H}_g((D+1)k, Ek; Z).$$

*Proof.* For brevity set  $t = \mathcal{H}_g((D+1)k, Ek; Z)$  and let  $P_1, \dots, P_t$  be a basis of  $I_{((D+1)k, Ek)}^\perp$ .

We can write  $W_2$  as

$$\{(G_1, G'_1, \dots, G_n, G'_n) \in R_{((D+1)k, Ek)}^{2n}; \langle G_i, P_j \rangle = \langle G'_i, P_j \rangle \text{ for } 1 \leq i \leq n, 1 \leq j \leq t\}.$$

Let  $A \in \text{Mat}_{N,t}(\overline{\mathbf{Q}})$  be the matrix whose columns are  $\iota(\tau(P_1)), \dots, \iota(\tau(P_t))$ ; here  $N = \binom{n+(D+1)k}{n} \binom{r+Ek}{r}$ . Then  $\iota(W_2)$  is the kernel of the rank  $nt$  matrix

$$B = \begin{pmatrix} A^\top & -A^\top & & 0 \\ & & \ddots & \\ 0 & & & A^\top & -A^\top \end{pmatrix} \in \text{Mat}_{nt, 2nN}(\overline{\mathbf{Q}}).$$

Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing the finitely many algebraic numbers which appear in this proof and let  $v \in M_K$ .

If  $v$  is infinite and  $\sigma = \sigma_v : K \rightarrow \mathbf{C}$  then the Cauchy-Binet formula implies  $\exp 2h_{v,nt}(B) = \det(\overline{\sigma(B)}\sigma(B)^\top)$ . We obtain a block-diagonal matrix

$$\overline{\sigma(B)}\sigma(B)^\top = \begin{pmatrix} 2\overline{\sigma(A)}^\top \sigma(A) & & 0 \\ & \ddots & \\ 0 & & 2\overline{\sigma(A)}^\top \sigma(A) \end{pmatrix}.$$

In total there are  $n$  blocks and so  $\det(\overline{\sigma(B)}\sigma(B)^\top) = 2^{nt} \det(\overline{\sigma(A)}^\top \sigma(A))^n$ . If we apply the Cauchy-Binet formula again we arrive at

$$(6.12) \quad h_{v,nt}(B) = nh_{v,t}(A) + \frac{nt}{2} \log 2.$$

Say  $v$  is finite. We fix a  $t \times t$  submatrix  $A'$  of  $A^\top$  with  $|\det A'|_v$  maximal. In particular,  $\det A' \neq 0$ . It follows that  $h_{v,t}(A'^{-1}A^\top) = h_{v,t}(A^\top) - \log |\det A'|_v = 0$ . We observe that the entries of  $A'^{-1}A^\top$  are  $v$ -integers. Consider the rank  $nt$  matrix

$$(6.13) \quad C = \underbrace{\begin{pmatrix} A'^{-1} & & 0 \\ & \ddots & \\ 0 & & A'^{-1} \end{pmatrix}}_{n \text{ blocks}} B = \begin{pmatrix} A'^{-1}A^\top & -A'^{-1}A^\top & & 0 \\ & & \ddots & \\ 0 & & A'^{-1}A^\top & -A'^{-1}A^\top \end{pmatrix}.$$

We have  $h_{v,nt}(C) \leq 0$ . The block matrix in the middle of (6.13) has determinant  $(\det A')^{-n}$ . This shows the equality in

$$h_{v,nt}(B) - n \log |\det A'|_v = h_{v,nt}(C) \leq 0.$$

We obtain

$$(6.14) \quad h_{v,nt}(B) \leq nh_{v,t}(A).$$

We multiply (6.12) and (6.14) with  $[K_v : \mathbf{Q}_v]/[K : \mathbf{Q}]$  and sum over all places to obtain

$$h_{nt}(B) \leq nh_t(A) + \frac{nt}{2} \log 2.$$

Now  $h_t(A)$  is the height of  $\iota(I_{((D+1)k, Ek)}^\perp)$ ; indeed, applying  $\tau$  does not change the height. So  $h_t(A) = \mathcal{H}_a((D+1)k, Ek; Z)$  by Lemma 6.4. The lemma follows because passing to the orthogonal complement (3.5) does not change the height; i.e.  $h_{\text{Ar}}(W_2) = h_{nt}(B)$ .  $\square$

We will not bound the height of  $W_3$  directly. Rather we construct a larger space of controlled height which, together with  $W_1$  and  $W_2$ , still cuts out  $W'_{DEk}$ .

**Lemma 6.7.** *There exists a vector subspace  $W'_3$  of  $R_{((D+1)k, Ek)}$  such that*

$$W'_{DEk} = W_1 \cap W_2 \cap W'_3$$

and

$$h_{\text{Ar}}(W'_3) \leq 20n \max\{n, r\}^2 \max\{D, E\} \mathcal{H}_g(k, Ek; Z)k.$$

*Proof.* For brevity we set  $M = \dim R_{((D+1)k, Ek)} = \binom{(D+1)k+n}{n} \binom{Ek+r}{r}$ . By the definition of  $W_3$  we see that  $\iota(W_3) \subset \overline{\mathbf{Q}}^{2nM}$  is cut out by  $nM$  linear equations; for each monomial in  $R_{((D+1)k, Ek)}$  we need  $n$  equations. And each linear equation comes from one equation

$$X_i^{Dk} G'_1 - X_1^{Dk} G'_i = 0.$$

With respect to the usual basis, a typical linear equations has coefficients

$$(6.15) \quad \binom{(D+1)k}{\alpha}^{1/2} \quad \text{and} \quad -\binom{(D+1)k}{\alpha'}^{1/2} \quad \text{with} \quad \alpha, \alpha' \in \mathbf{N}_0^{n+1} \quad \text{and} \quad |\alpha| = |\alpha'| = (D+1)k.$$

Among these  $nM$  linear equations we can find  $t \leq \dim W_1$  linearly independent ones such that if  $\iota(W'_3)$  is their common kernel then  $W'_{DEk} = W_1 \cap W_2 \cap W'_3$ . Let  $A$  be a  $t \times 2nM$  matrix whose rows are precisely these chosen linear equations. Hence the kernel of  $A$  is  $\iota(W'_3)$  and each row has at most two non-zero entries of the form (6.15).

Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing the finitely many algebraic numbers which appear in this proof and let  $v \in M_K$ .

Say  $v$  is infinite. The absolute value of the multinomials in (6.15) is at most  $(n+1)^{(D+1)k/2}$ . Let  $A'$  be a  $t \times t$  submatrix of  $A$ . By the discussion above we obtain

$$|\det A'|_v \leq (2(n+1)^{(D+1)k/2})^t$$

from the Leibniz formula for the determinant. Now the number of possible  $t \times t$  submatrices of  $A$  is

$$\binom{2nM}{t} \leq (2nM)^t = (2n)^t \binom{(D+1)k+n}{n}^t \binom{Ek+r}{r}^t \leq (2n)^t ((D+1)k+n)^{nt} (Ek+r)^{rt}.$$

By definition of the local height of  $A$  we get

$$(6.16) \quad h_{v,t}(A) \leq \frac{1}{2}tk(D+1)\log(n+1) + t\log 2 + \frac{t}{2}(\log(2n) + n\log((D+1)k+n) + r\log(Ek+r)).$$

If  $v$  is finite then  $h_{v,t}(A) \leq 0$  since the coefficients of  $A$  are algebraic integers.

We use this observation, multiply (6.16) with  $[K_v : \mathbf{Q}_v]/[K : \mathbf{Q}]$ , and sum over all places to obtain

$$h_t(A) \leq \frac{1}{2}tk(D+1)\log(n+1) + t\log 2 + \frac{t}{2}(\log(2n) + n\log((D+1)k+n) + r\log(Ek+r)).$$

Now  $t \leq \dim W_1 = 2n \dim I_{(k, Ek)}^\perp = 2n\mathcal{H}_g(k, Ek; Z)$ . So

$$h_t(A) \leq (nk(D+1)\log(n+1) + n(\log(8n) + n\log((D+1)k+n) + r\log(Ek+r)))\mathcal{H}_g(k, Ek; Z).$$

Again, height invariance under passing to the orthogonal complement gives  $h_{\text{Ar}}(W'_3) = h_t(A)$ . The lemma follows from the following elementary inequalities

$$\begin{aligned} nk(D+1)\log(n+1) + n(\log(8n) + n\log((D+1)k+n) + r\log(Ek+r)) \\ \leq 2n^2Dk + 8n^2 + 2n^2Dk + n^3 + nrEk + nr^2 \\ \leq 20n \max\{n, r\}^2 \max\{D, E\}k. \quad \square \end{aligned}$$



We need precise estimates for the arithmetic Hilbert function  $\mathcal{H}_a(ak, bk; Z)$  at large values  $k$ . Our tools are Proposition 4.1 and Zhang's inequality [48] for the essential minimum for a subvariety of  $\mathbf{P}^n$ . We recall that  $s$  and  $v_{ab}$  denote the Segre and Veronese morphism, respectively. Also, degree and height of a subvariety of  $\mathbf{P}^n \times \mathbf{P}^r$  is the degree and height of its embedding into  $\mathbf{P}^{nr+n+r}$  under the Segre morphism, respectively.

**Lemma 6.8.** *We have*

$$h(v_{ab}(Z)) \leq (1 + d) \max\{a, b\}^{d+1} h(Z)$$

for  $a, b \in \mathbf{N}$ .

*Proof.* If  $X \subset \mathbf{P}^n$  is an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$ , then its essential minimum is

$$\mu^{\text{ess}}(X) = \inf \{ \theta \in \mathbf{R}; \{z \in X(\overline{\mathbf{Q}}); h(z) \leq \theta\} \text{ is Zariski dense in } X \}.$$

Let  $\epsilon > 0$ , by definition

$$\{z \in s(Z)(\overline{\mathbf{Q}}); h(z) \leq \mu^{\text{ess}}(s(Z)) + \epsilon\} \text{ is Zariski dense in } s(Z).$$

Any  $z$  in this set is of the form  $s(p, q)$  with  $(p, q) \in Z(\overline{\mathbf{Q}})$ . By Section 1.5.14 [4] we have  $h(z) = h(p) + h(q)$ . An elementary local estimate shows  $h(v_a(p)) \leq ah(p)$  and  $h(v_b(q)) \leq bh(q)$ . We have  $v_{ab}(z) = (v_a(p), v_b(q))$ , so  $h(s(v_{ab}(z))) = h(v_a(p)) + h(v_b(q)) \leq ah(p) + bh(q) \leq \max\{a, b\}h(z)$ . The set of  $s(v_{ab}(z))$  thus obtained is Zariski dense in  $s(v_{ab}(Z))$ . We get  $\mu^{\text{ess}}(s(v_{ab}(Z))) \leq \max\{a, b\}(\mu^{\text{ess}}(s(Z)) + \epsilon)$  for all  $\epsilon > 0$ . Letting  $\epsilon$  go to zero gives

$$(6.17) \quad \mu^{\text{ess}}(s(v_{ab}(Z))) \leq \max\{a, b\} \mu^{\text{ess}}(s(Z)).$$

Zhang's inequality states

$$(6.18) \quad \frac{h(X)}{(1 + \dim X) \deg(X)} \leq \mu^{\text{ess}}(X) \leq \frac{h(X)}{\deg(X)}.$$

We use his inequality to bound  $\mu^{\text{ess}}(s(v_{ab}(Z)))$  from below and  $\mu^{\text{ess}}(s(Z))$  from above. Inequality (6.17) implies

$$(6.19) \quad \frac{h(v_{ab}(Z))}{(1 + d) \deg(v_{ab}(Z))} \leq \mu^{\text{ess}}(s(v_{ab}(Z))) \leq \max\{a, b\} \mu^{\text{ess}}(s(Z)) \leq \max\{a, b\} \frac{h(Z)}{\deg(Z)};$$

we note  $\dim s(v_{ab}(Z)) = d$ . Lemmas 4.1 and 4.2 give  $\deg(v_{ab}(Z)) = H(1, 1; v_{ab}(Z)) = H(a, b; Z) \leq \max\{a, b\}^d \deg(Z)$ . The current lemma follows from (6.19).  $\square$

**Lemma 6.9.** *If  $k \geq \deg(Z)$ , then*

$$\mathcal{H}_a(ak, bk; Z) \leq 2r \max\{a, b\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k$$

for  $a, b \in \mathbf{N}$ .

*Proof.* By Lemma 5.5 we deduce  $\mathcal{H}_a(ak, bk; Z) \leq \mathcal{H}_a(k, k; v_{ab}(Z))$ . For brevity, we set  $X = s(v_{ab}(Z))$  and use Lemma 5.4 to estimate  $\mathcal{H}_a(k, k; v_{ab}(Z)) \leq \mathcal{H}_a(k; X)$ . Hence

$$\mathcal{H}_a(ak, bk; Z) \leq \mathcal{H}_a(k; X).$$

We apply Proposition 4.1 to bound the arithmetic Hilbert function of  $X$  and obtain

$$(6.20) \quad \mathcal{H}_a(ak, bk; Z) \leq \mathcal{H}_g(k; X) \left( k \frac{h(X)}{\deg(X)} + \frac{1}{2} \log \mathcal{H}_g(k; X) \right).$$

We continue by bounding the height of  $X$ . Lemma 6.8 implies

$$h(X) \leq (d+1) \max\{a, b\}^{d+1} h(Z).$$

We insert this into (6.20) to get

$$(6.21) \quad \mathcal{H}_a(ak, bk; X) \leq (d+1) \max\{a, b\}^{d+1} k \frac{\mathcal{H}_g(k; X)}{\deg(X)} h(Z) + \frac{1}{2} \mathcal{H}_g(k; X) \log \mathcal{H}_g(k; X).$$

Chardin's bound for the geometric Hilbert function [12] states

$$\mathcal{H}_g(k; X) \leq \deg(X) \binom{d+k}{d}.$$

Hence

$$(6.22) \quad \mathcal{H}_a(ak, bk; X) \leq (d+1) \max\{a, b\}^{d+1} h(Z) \binom{d+k}{d} k + \frac{1}{2} \deg(X) \binom{d+k}{d} \log \mathcal{H}_g(k; X).$$

Lemmas 4.1 and 4.2 give  $\deg(X) = \deg(v_{ab}(Z)) = H(a, b; Z) \leq \max\{a, b\}^d \deg(Z)$ . Elementary estimates imply  $\binom{d+k}{d} \leq (d+k)^d/d! \leq (d+1)^d k^d/d! \leq (ek)^d$ . Hence  $\log \mathcal{H}_g(k; X) \leq \log(\deg(X)(ek)^d) \leq d \log(e \max\{a, b\} \deg(Z) k)$ . We recall  $k \geq \deg(Z)$  to see that

$$\log \mathcal{H}_g(k; X) \leq d \log(e \max\{a, b\} k^2) \leq 2d \log(e \max\{a, b\} k) \leq 2d \max\{a, b\} k.$$

We apply this and the bound for  $\deg(X)$  from above to (6.22) and obtain

$$\mathcal{H}_a(ak, bk; X) \leq (d+1) \max\{a, b\}^{d+1} h(Z) \binom{d+k}{d} k + d \max\{a, b\}^{d+1} \deg(Z) \binom{d+k}{d} k.$$

The lemma follows since  $d+1 \leq 2d \leq 2r$ .  $\square$

We now bound the height of  $W'_{DEk}$  from above explicitly in terms of  $h(Z)$ ,  $\deg(Z)$ , and the parameters  $D$ ,  $E$ , and  $k$ .

**Lemma 6.10.** *If  $k \geq \deg(Z)$ , then*

$$h_{\text{Ar}}(W_1) \leq 24n \max\{n, r\}^2 \max\{D, E\} E^d (h(Z) + \deg(Z)) \binom{d+k}{d} k,$$

$$h_{\text{Ar}}(W_2) \leq 3nr \max\{D+1, E\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k, \text{ and}$$

$$h_{\text{Ar}}(W'_3) \leq 20n \max\{n, r\}^2 \max\{D, E\} E^d \deg(Z) \binom{d+k}{d} k.$$

*Proof.* The inequalities

$$(6.23) \quad \mathcal{H}_a(k, Ek; Z) \leq 2r E^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k$$

$$\mathcal{H}_g(k, Ek; Z) \leq H(1, E; Z) \binom{d+k}{d}$$

follow from Lemmas 6.9 and 4.4, respectively. Lemma 4.1 gives the bound  $H(1, E; Z) \leq E^d \deg(Z)$ , so

$$(6.24) \quad \mathcal{H}_g(k, Ek; Z) \leq E^d \deg(Z) \binom{d+k}{d}.$$

The same lemmas and Lemma 4.1 imply

$$(6.25) \quad \mathcal{H}_a((D+1)k, Ek; Z) \leq 2r \max\{D+1, E\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k,$$

$$(6.26) \quad \mathcal{H}_g((D+1)k, Ek; Z) \leq H(D+1, E; Z) \binom{d+k}{d} \leq \max\{D+1, E\}^d \deg(Z) \binom{d+k}{d}.$$

First, we bound  $h_{\text{Ar}}(W_1)$ . By the estimate in Lemma 6.5 together with (6.23) and (6.24) we get

$$\begin{aligned} h_{\text{Ar}}(W_1) &\leq 4nr E^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k + 20n \max\{n, r\}^2 \max\{D, E\} E^d \deg(Z) \binom{d+k}{d} k \\ &\leq (4nr + 20n \max\{n, r\}^2) \max\{D, E\} E^d (h(Z) + \deg(Z)) \binom{d+k}{d} k \\ &\leq 24n \max\{n, r\}^2 \max\{D, E\} E^d (h(Z) + \deg(Z)) \binom{d+k}{d} k. \end{aligned}$$

This is the bound for  $h_{\text{Ar}}(W_1)$  in the lemma.

Next, we bound  $h_{\text{Ar}}(W_2)$ . For this we need Lemma 6.6 which, together with the bounds (6.25) and (6.26), gives

$$\begin{aligned} h_{\text{Ar}}(W_2) &\leq 2nr \max\{D+1, E\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k + n \max\{D+1, E\}^d \deg(Z) \binom{d+k}{d} k \\ &\leq 3nr \max\{D+1, E\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k \end{aligned}$$

and so the second bound in the assertion.

Finally, we give a bound for  $h_{\text{Ar}}(W'_3)$ . Applying Lemma 6.7 and (6.24) leads to the final bound

$$h_{\text{Ar}}(W'_3) \leq 20n \max\{n, r\}^2 \max\{D, E\} E^d \deg(Z) \binom{d+k}{d} k. \quad \square$$

By Lemmy 6.7, the vector space  $W'_{DEk}$  is the intersection  $W_1 \cap W_2 \cap W'_3$ , this enables us to bound its height.

**Lemma 6.11.** *If  $k \geq \deg(Z)$ , then*

$$h_{\text{Ar}}(W'_{DEk}) \leq 50n \max\{n, r\}^2 \max\{D+1, E\}^{d+1} (h(Z) + \deg(Z)) \binom{d+k}{d} k.$$

*Proof.* A theorem of Schmidt, cf. (3.4) in Remark 3.4, implies  $h_{\text{Ar}}(W'_{DEk}) \leq h_{\text{Ar}}(W_1) + h_{\text{Ar}}(W_2) + h_{\text{Ar}}(W'_3)$ . Adding the bounds provided by the previous lemma leads to the desired inequality.  $\square$

**6.4. Proof of Propositions 6.1 and 6.2.** We continue working with the notation introduced in the preceding subsections. We recall that  $\kappa(Z)$  was defined in (6.2).

We will need some preparatory estimates.

**Lemma 6.12.** *We have*

$$(6.27) \quad \kappa(Z) \leq \deg(Z).$$

*Proof.* Indeed, Lemma 4.1 and (6.1) imply  $\deg(Z) = \sum_{i=0}^d \binom{d}{i} \Delta_i(Z) \geq \Delta_0(Z)$ . We may assume  $\kappa(Z) \geq 1$ . By definition there is  $1 \leq i \leq d$  with  $\Delta_i(Z) \neq 0$  and  $\kappa(Z) = (\Delta_0(Z)/\Delta_i(Z))^{1/i} \leq \Delta_0(Z)/\Delta_i(Z) \leq \Delta_0(Z) \leq \deg(Z)$  and our lemma holds.  $\square$

The next lemma bounds the value of  $\Psi$ , defined near (6.9), from above.

**Lemma 6.13.** *Let  $G = (G_1, G'_1, \dots, G_n, G'_n) \in \prod_{i=1}^n V_0 \times V_i$  be non-zero, then  $h(\Psi(G)) \leq h(G) + (\frac{D+1}{2} \log(n+1) + n+1)k$ .*

*Proof.* By the convention introduced in Section 3.2, the height of  $G$  is the height of  $\iota(G)$ .

Tracing through the definition of  $\iota$  we see that each coordinate of  $\iota(\Psi(G))$  is some coordinate of  $\iota(G)$  times a factor of the form

$$(6.28) \quad \binom{(D+1)k}{\alpha}^{1/2} \binom{k}{\alpha'}^{-1/2} \quad \text{with } \alpha, \alpha' \in \mathbf{N}_0^{n+1} \text{ and } |\alpha|_1 = (D+1)k, |\alpha'|_1 = k.$$

Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing all algebraic numbers which appear below and let  $v \in M_K$ .

Say  $v$  is infinite. Then the absolute value of the expression (6.28) with respect to  $v$  is bounded by  $(n+1)^{(D+1)k/2}$ . It follows that

$$(6.29) \quad |\Psi(G)|_v \leq (n+1)^{(D+1)k/2} |G|_v$$

Now say if  $v$  is finite and let  $p$  be the rational prime with  $|p|_v < 1$ . The  $v$ -adic absolute value of (6.28) is at most  $k^{(n+1)/2}$  if  $p \leq k$  and at most 1 if  $p > k$ , cf. (3.1). It follows that

$$(6.30) \quad \log |\Psi(G)|_v \leq \log |G|_v + \begin{cases} \frac{n+1}{2} \log k & : \text{if } p \leq k, \\ 0 & : \text{else wise.} \end{cases}$$

Multiplying the expressions (6.29) and (6.30) with the corresponding local degrees and taking the sum over all places gives

$$h(\Psi(G)) \leq h(G) + \frac{D+1}{2} k \log(n+1) + \frac{n+1}{2} \pi(k) \log k$$

here, as in the proof of Lemma 6.5,  $\pi(k)$  denotes the number of rational primes at most  $k$ . We have  $\pi(k) \log k \leq 2k$  and this completes the proof.  $\square$

The following absolute version of Siegel's Lemma is due to Zhang. We could have also referred to the version of Roy and Thunder [42].

**Lemma 6.14.** *Suppose  $\dim W'_{DEk} \geq 2$ . There exists a non-zero  $G \in W'_{DEk}$  such that*

$$h(G) \leq \frac{h_{\text{Ar}}(W'_{DEk})}{\dim W'_{DEk}} + \log \dim W'_{DEk}.$$

*Proof.* This is a consequence of David and Philippon's Lemme 4.7 [15] which is based on a result of Zhang. We added the artificial hypothesis  $\dim W'_{DEk} \geq 2$  to avoid the  $\epsilon$  in the reference.  $\square$

This variant of Siegel's Lemma is needed in the next lemma.

**Lemma 6.15.** *We assume*

$$\frac{D+1}{E} \leq \frac{1}{4dn} \kappa(Z)$$

and

$$k \geq \max\{[17nd!e^d \Delta_0^{d-1}], \deg(Z)\}.$$

There exists a non-zero  $F \in W_{DEk}$  such that

$$\frac{1}{Ek} h(F) \leq 400n \max\{n, r\}^2 \frac{\max\{1, \kappa(Z)\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 5d \deg(Z).$$

*Proof.* Recall that  $\Psi$  given by (6.9) defines an isomorphism between  $W'_{DEk}$  and  $W_{DEk}$ .

Let us assume that  $k$  is as in the hypothesis. Then  $k \geq 16nd!e^d \Delta_0^{d-1}$  and so

$$\frac{1}{2} \frac{k^d}{d!} - 4ne^d \Delta_0^{d-1} k^{d-1} \geq \frac{1}{4} \frac{k^d}{d!}.$$

Lemma 6.3 implies

$$\dim W'_{DEk} = \dim W_{DEk} \geq \frac{1}{4} \Delta_0 E^d \frac{k^d}{d!};$$

this implies  $\dim W'_{DEk} \geq 2$  by our choice of  $k$ .

An elementary calculation shows  $\binom{d+k}{d} \leq 2k^d/d!$  since  $k \geq 16d!$ . So

$$\dim W'_{DEk} \geq \frac{1}{8} \Delta_0 E^d \binom{d+k}{d}.$$

Because of Lemmas 6.14 and 6.11 there exists a non-zero  $G \in W'_{DEk}$  with

$$\begin{aligned} h(G) &\leq \frac{h_{\text{Ar}}(W'_{DEk})}{\dim W'_{DEk}} + \log \dim W'_{DEk} \\ &\leq 400n \max\{n, r\}^2 \frac{\max\{D+1, E\}^{d+1}}{\Delta_0 E^d} (h(Z) + \deg(Z))k + \log \dim W'_{DEk}. \end{aligned}$$

We need to bound  $\dim W'_{DEk} = \dim W_{DEk}$  from above in order to get a handle on the logarithm. Indeed, by definition (6.5) we have  $\dim W'_{DEk} \leq (n+1) \mathcal{H}_g(k, Ek; Z)$ . Just as around (6.24) in the proof of Lemma 6.10 we have  $\mathcal{H}_g(k, Ek; Z) \leq E^d \deg(Z) \binom{d+k}{d} \leq 2E^d \deg(Z) k^d/d!$ , so  $\log \dim W'_{DEk} \leq \log(4nE^d \deg(Z) k^d) \leq d \log(4nE \deg(Z) k)$ . We obtain

$$\frac{d \log(4nE \deg(Z) k)}{Ek} = \frac{d \log(4n)}{Ek} + \frac{d \log E}{Ek} + \frac{d \log \deg(Z)}{Ek} + \frac{d \log k}{Ek} \leq 1 + 1 + d + d \leq 4d$$

from  $k \geq 16dn$  and  $k \geq \deg(Z)$ . Thus

$$\log \dim W_{DEk} \leq 4dEk.$$

We set  $F = \Psi(G) \neq 0$ . Its height is bounded above by Lemma 6.13, we obtain the estimate

$$\begin{aligned} h(F) &\leq h(G) + \left( \frac{D+1}{2} \log(n+1) + n+1 \right) k \leq h(G) + 3nDk \\ &\leq 400n \max\{n, r\}^2 \frac{\max\{D+1, E\}^{d+1}}{\Delta_0 E^d} (h(Z) + \deg(Z))k + 4dEk + 3nDk. \end{aligned}$$

We use the bound  $(D+1)/E \leq \kappa(Z)/(4dn) \leq \kappa(Z)$  to estimate

$$\frac{1}{Ek} h(F) \leq 400n \max\{n, r\}^2 \frac{\max\{1, \kappa(Z)\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 4d + \kappa(Z).$$

Inequality (6.27) implies  $4d + \kappa(Z) \leq 5d\deg(Z)$  and the lemma follows.  $\square$

*Proof of Proposition 6.1.* In order to prove the proposition we may assume that no projective coordinate  $X_i$  vanishes identically on  $Z$ . Indeed, otherwise we are in case (i).

Let us assume for the moment that  $\kappa < 17dn$ . For  $Q \geq 1$  there are integers  $x, y \in \mathbf{Z}$  with  $1 \leq y \leq Q$  such that  $|y\kappa/(8dn) - x| \leq Q^{-1}$ ; this follows easily from Dirichlet's Box Principle as employed on the first page of Cassels' book [11].

We take  $Q = \max\{1, 16dn/\kappa\}$ . Then  $|x/y| \leq \kappa/(8dn) + 1/(yQ) \leq \kappa/(8dn) + \kappa/(16dn) \leq \kappa/(4dn)$  and  $|x/y| \geq \kappa/(8dn) - 1/(yQ) \geq \kappa/(8dn) - \kappa/(16dn) = \kappa/(16dn)$ . Finally,  $x \geq 1$ ; indeed, otherwise we would have  $y\kappa/(8dn) - x \geq \kappa/(8dn) > Q^{-1}$  and this is a contradiction.

We set  $D = 4x - 1$  and  $E = 4y$ . Hence  $D, E$  are positive integers with

$$(6.31) \quad D \geq 3, \quad E \leq 4 \max\{1, 16dn/\kappa\}, \quad \text{and} \quad \frac{\kappa}{16dn} \leq \frac{D+1}{E} \leq \frac{\kappa}{4dn}.$$

If  $\kappa \geq 17dn$ , we set  $D = \lceil \kappa/(4dn) \rceil - 1$  and  $E = 1$ . So they also satisfy all inequalities in (6.31).

We pick  $k = \max\{[17nd!e^d \Delta_0^{d-1}], \deg(Z)\}$  in accordance with Lemma 6.15. Let  $F = (F_0, \dots, F_n) \in W_{DEk}$  be as provided by this lemma. We note that

$$Ek \leq k_0.$$

Some  $F_i$  is non-zero. As an element of  $I_{(k, Ek)}^\perp$ , it does not vanish identically on  $Z$  by what was stated in Remark 6.1. We have assumed that  $X_0$  does not vanish identically on  $Z$ . It follows from (6.5), the definition of  $W_{DEk}$ , that  $F_0$  does not vanish identically on  $Z$ . We take  $F$  in the assertion of the current proposition to be  $F_0$ . Note that it is bihomogeneous of bidegree  $(a, b) = (k, Ek)$ .

The desired bounds for  $a$  and  $b$  follow from our choice of  $k$ . Moreover, if  $\kappa \geq 17dn$ , then  $E = 1$  and so  $a = b$ . The bound for  $h(F_0)$  follows from  $h(F_0) \leq h(F) = h(F_0, \dots, F_n)$ , from our choice of  $k$ , and from Lemma 6.15.

Say  $(p, q) \in Z(\overline{\mathbf{Q}})$  and let us assume that we are not in case (i) of the proposition. We are in the position to apply Lemma 6.1 with  $(a, b)$  as above and  $c = Dk$ . We obtain

$$\frac{D-1}{E} h(p) \leq h(q) + \frac{1}{Ek} h(F) + \frac{D}{2E} \log(n+1) \leq h(q) + \frac{1}{Ek} h(F) + \frac{\kappa}{8d}$$

where we used  $D/E \leq \kappa/(4dn)$  from (6.31) and  $\log(n+1) \leq n$ .



The bound for  $h(F)$  given by Lemma 6.15 implies

$$\frac{D-1}{E}h(p) \leq h(q) + 400n \max\{n, r\}^2 \frac{\max\{1, \kappa\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 5d \deg(Z) + \frac{\kappa}{8d}.$$

Since  $D \geq 3$  we have  $(D-1)/E \geq (D+1)/(2E)$  and so  $(D-1)/E \geq \kappa/(32dn)$  by (6.31). Hence

$$\kappa h(p) \leq 2^5 d n h(q) + 12800 d n^2 \max\{n, r\}^2 \frac{\max\{1, \kappa\}^{d+1}}{\Delta_0} (h(Z) + \deg(Z)) + 160 d^2 n \deg(Z) + 4n\kappa.$$

The inequality in part (ii) of the assertion follows from  $d \leq r$  and  $\kappa \leq \deg(Z)$ , cf. (6.27).  $\square$

**Lemma 6.16.** *Let  $a \in \mathbf{N}$  and say  $P \in \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,a)}$ . There is  $Q \in \overline{\mathbf{Q}}[\mathbf{U}]_a$  with  $s^*(Q) = P$  and*

$$h(Q) \leq h(P) + (n + r + 2)a.$$

*Proof.* Just for this proof we let  $\langle \cdot, \cdot \rangle$  be the inner product attached to the identity on  $\overline{\mathbf{Q}}$  as in Remark 5.2.

We know that  $s^* : \overline{\mathbf{Q}}[\mathbf{U}]_a \rightarrow \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,a)}$  is a surjective isometry by Lemma 5.2. It follows that  $\ker s^* \cap (\ker s^*)^\perp = 0$  by Remark 5.1. So  $\ker s^* + (\ker s^*)^\perp = \overline{\mathbf{Q}}[\mathbf{U}]_a$ . Therefore,  $s^*|_{(\ker s^*)^\perp} : (\ker s^*)^\perp \rightarrow \overline{\mathbf{Q}}[\mathbf{X}, \mathbf{Y}]_{(a,a)}$  is surjective. There is  $Q \in (\ker s^*)^\perp$  with  $s^*(Q) = P$ . We write  $Q = \sum_\gamma Q_\gamma \mathbf{U}^\gamma$  where  $\gamma$  runs over elements in  $\mathbf{N}_0^{(n+1)(r+1)}$  with  $|\gamma|_1 = a$ . It remains to bound the height of  $Q$  from above.

Let  $K \subset \overline{\mathbf{Q}}$  be a finite normal extension of  $\mathbf{Q}$  containing all of the finitely many algebraic numbers that appear in the proof below. Let  $v \in M_K$ .

If  $v$  is finite, there is  $\gamma_0 \in \mathbf{N}_0^{(n+1)(r+1)}$  with  $|Q|_v = \left| \binom{a}{\gamma_0}^{-1/2} Q_{\gamma_0} \right|_v$ . We have  $\binom{a}{\gamma_0}^{-1} Q_{\gamma_0} = \langle Q, \mathbf{U}^{\gamma_0} \rangle = \langle P, s^*(\mathbf{U}^{\gamma_0}) \rangle$  since  $s^*$  is an isometry. This inner product equals  $\binom{a}{\alpha}^{-1} \binom{a}{\beta}^{-1} P_{\alpha\beta}$  with  $\alpha \in \mathbf{N}_0^{n+1}$  and  $\beta \in \mathbf{N}_0^{r+1}$  and where  $P_{\alpha\beta}$  is a coefficient of  $P$ . Hence

$$|Q|_v = \left| \binom{a}{\gamma_0}^{-1/2} Q_{\gamma_0} \right|_v = \left| \binom{a}{\gamma_0}^{1/2} \binom{a}{\alpha}^{-1} \binom{a}{\beta}^{-1} P_{\alpha\beta} \right|_v \leq \left| \binom{a}{\alpha} \binom{a}{\beta} \right|_v^{-1/2} |P|_v.$$

Together with (3.1) we have

$$(6.32) \quad |Q|_v \leq |P|_v \cdot \begin{cases} a^{(n+r+2)/2} & : \text{ if } p \leq a, \\ 1 & : \text{ else wise } \end{cases}$$

where  $p$  is the rational prime with  $|p|_v < 1$ .

Recall that  $\tau$  is complex conjugation restricted to  $\overline{\mathbf{Q}}$ . If  $v$  is infinite and  $\sigma = \sigma_v$ , then there is an automorphism  $\eta$  of  $K$  such that  $\sigma \circ \eta = \tau \circ \sigma$ . We have  $|Q|_v^2 = \iota(\sigma(Q))^\top \cdot \tau(\iota(\sigma(Q))) = \sigma(\langle Q, \eta(Q) \rangle)$ . But  $Q \in \ker s^{*\perp}$  and  $s^*$  is an isometry, so  $|Q|_v^2 = \sigma(\langle s^*(Q), s^*(\eta(Q)) \rangle) = \sigma(\langle s^*Q, \eta(s^*Q) \rangle) = \sigma(\langle P, \eta(P) \rangle) = |P|_v^2$ . Therefore,

$$(6.33) \quad |Q|_v = |P|_v.$$

We multiply the logarithm of (6.32) and (6.33) with  $[K_v : \mathbf{Q}_v]/[K : \mathbf{Q}]$  and sum over all places of  $K$  to obtain

$$h(Q) \leq h(P) + \frac{n + r + 2}{2} \pi(a) \log a$$

where  $\pi(a)$  is the number of rational primes at most  $a$ . The lemma follows from  $\pi(a) \log a \leq 2a$ , an inequality we have already seen twice.  $\square$

We now prove Proposition 6.2.

For brevity set  $\beta(Z) = h(Z) + \deg(Z)$ .

Let  $a, b$ , and  $F$  be as in Proposition 6.1; we note that  $b = a$  since  $\kappa \geq 17dn$  by hypothesis.

Say  $(p, q)$  is as in the proposition. If some projective coordinate of  $p$  vanishes or if  $p$  is not isolated in  $\pi_1|_Z^{-1}(p)$ , then we are in case (i). So let us assume the contrary.

If  $F(p, q) \neq 0$  then we are in case (ii) of Proposition 6.1. The height bound for  $h(p)$  implies

$$\begin{aligned} \kappa h(p) &\leq 2^5 dn h(q) + 2^{14} n^2 r \max\{n, r\}^2 \frac{\kappa^{d+1}}{\Delta_0(Z)} \beta(Z) + 2^8 n r^2 \deg(Z) \\ &\leq 2^5 dn h(q) + 2^{14} n^2 r \max\{n, r\}^2 \left( \frac{\kappa^{d+1}}{\Delta_0(Z)} + 1 \right) \beta(Z). \end{aligned}$$

The inequality in part (iii) now follows easily.

It remains to treat the case  $F(p, q) = 0$ . Then  $F$  will determine the obstruction variety  $V$  as follows. Let  $\tilde{V}_1, \dots, \tilde{V}_N$  be the irreducible components of the intersection of  $Z$  with the zero set of  $F$ . Of course, these are independent of  $(p, q)$ . Then  $\dim \tilde{V}_i = d - 1$  because  $F$  does not vanish identically on  $Z$ . We may omit those  $V_i$  for which  $\pi_1|_{\tilde{V}_i}$  has degree zero since  $(p, q)$  is isolated in  $\pi_1^{-1}|_Z(p)$ . Let us set  $V_i = \pi_1(\tilde{V}_i)$ . Our point  $p$  is contained in some  $V_i$ .

By the Fiber Dimension Theorem,  $V_i$  has dimension  $d - 1$ .

We apply Lemma 6.16 to obtain  $Q \in \overline{\mathbf{Q}}[\mathbf{U}]_a$  with  $s^*(Q) = F$  and

$$(6.34) \quad h(Q) \leq h(F) + (n + r + 2)a.$$

The images  $s(\tilde{V}_i)$  are irreducible components of the intersection of  $s(Z)$  with the zero-set of  $Q$ . By Bézout's Theorem, cf. Example 8.4.6 [17], we estimate  $\sum_{i=1}^N \deg(\tilde{V}_i) = \sum_{i=1}^N \deg(s(\tilde{V}_i)) \leq a \deg(s(Z)) = a \deg(Z)$ .

By Lemma 4.1 we have  $\Delta_{d-1}(\tilde{V}_i) \leq \deg(\tilde{V}_i)$  for  $1 \leq i \leq N$ . The projection formula implies

$$(6.35) \quad \deg(V_i) = \deg(\pi(\tilde{V}_i)) \leq \Delta_{d-1}(\tilde{V}_i) \leq \deg(\tilde{V}_i)$$

so

$$\sum_{i=1}^N \deg(V_i) \leq \sum_{i=1}^N \deg(\tilde{V}_i) \leq a \deg(Z)$$

The bound for  $a$  from Proposition 6.1 leads to the bound (6.3).

It remains to bound each  $h(V_i)$  from above. For brevity set  $V = V_i$  and  $\tilde{V} = \tilde{V}_i$  for some valid  $i$ .

Let  $K \subset \overline{\mathbf{Q}}$  be a number field containing all of the finitely many algebraic numbers that appear in the proof below and let  $v \in M_K$ .

If  $v$  is infinite, then the Cauchy-Schwarz inequality implies  $|\sigma_v(Q)(x)| \leq |\iota(Q)|_v |x|_v^{\deg(Q)}$  for all  $x \in \mathbf{C}^{(n+1)(r+1)}$ . Therefore,

$$\sum_{v \in M_K} \frac{[K_v : \mathbf{Q}_v]}{[K : \mathbf{Q}]} m_v(Q) \leq h(Q) + \deg(Q) \sum_{j=1}^{nr+n+r} \frac{1}{2j}.$$

with  $m_v(Q)$  as in Section 3.3. We recall  $\deg(Q) = a$ . If  $W$  is the hypersurface defined by  $Q$  in  $\mathbf{P}^n$ , then  $\deg(W) = a$  and  $h(W) \leq h(Q) + a \sum_{i=1}^{nr+n+r} \sum_{j=1}^i \frac{1}{2j}$  by the comment on the top of page 347 [31]. We deduce

(6.36)

$$\begin{aligned} h(W) &\leq h(Q) + \frac{a}{2} \sum_{i=1}^{nr+n+r} (1 + \log i) \leq h(Q) + \frac{a}{2} (n+1)(r+1) \log((n+1)(r+1)) \\ &\leq h(Q) + 2anr \log(4nr) \leq h(F) + (n+r+2)a + 2anr \log(4nr) \\ &\leq h(F) + 6anr \log(4nr) \end{aligned}$$

using the bound for  $h(Q)$  from (6.34). Since  $s(\tilde{V})$  is an irreducible component of  $s(Z) \cap W$  we have

$$h(\tilde{V}) = h(s(\tilde{V})) \leq ah(Z) + h(W)\deg(Z) + c\deg(Z)$$

by the Arithmetic Bézout Theorem, Théorème 3 [31]; here

$$\begin{aligned} c &\leq (n+1)(r+1) \log 2 + \sum_{i=0}^{nr+n+r} \sum_{j=0}^{nr+n+r} \frac{1}{2(i+j+1)} \\ &\leq (n+1)(r+1) \log 2 + \frac{1}{2} (n+1)(r+1) \sum_{j=1}^{(n+1)(r+1)} \frac{1}{j}. \end{aligned}$$

This implies

$$(6.37) \quad h(\tilde{V}) \leq ah(Z) + h(W)\deg(Z) + 8anr \log(4nr)\deg(Z).$$

By definition, for any  $\epsilon > 0$  there is a Zariski dense set of points  $(p', q') \in \tilde{V}(\overline{\mathbf{Q}})$  with  $h(p') + h(q') = h(s(p', q')) \leq \mu^{\text{ess}}(s(\tilde{V})) + \epsilon$ . The resulting set of  $p'$  lies Zariski dense in  $V$ , hence  $\mu^{\text{ess}}(V) \leq \mu^{\text{ess}}(s(\tilde{V}))$  after letting  $\epsilon$  go to 0. Zhang's inequality (6.18) implies  $h(V) \leq (\dim V + 1) \frac{\deg(V)}{\deg(\tilde{V})} h(\tilde{V}) \leq d \frac{\deg(V)}{\deg(\tilde{V})} h(\tilde{V})$ . Using (6.35) we get  $h(V) \leq dh(\tilde{V})$ . Moreover, (6.37) gives

$$\begin{aligned} h(V) &\leq adh(Z) + dh(W)\deg(Z) + 8adnr \log(4nr)\deg(Z) \\ &\leq 14adnr \log(4nr)\beta(Z) + dh(F)\deg(Z). \end{aligned}$$

where we used (6.36) to bound  $h(W)$  in terms of  $h(F)$ . The bound for  $h(F)$  and the inequalities  $\kappa \geq 1$  and  $a \leq k_0$  show

(6.38)

$$\begin{aligned} h(V) &\leq 14dnr \log(4nr) \beta(Z) k_0 + d \left( 400n \max\{n, r\}^2 \frac{\kappa^{d+1}}{\Delta_0} \beta(Z) + 5d \deg(Z) \right) \deg(Z) k_0 \\ &\leq d \left( 14nr \log(4nr) + 400n \max\{n, r\}^2 \frac{\kappa^{d+1}}{\Delta_0} \deg(Z) + 5d \deg(Z) \right) \beta(Z) k_0. \end{aligned}$$

We estimate  $\log(4nr) = \log(2n) + \log(2r) \leq n + r \leq 2 \max\{n, r\}$  and recall  $d \leq r$ . Inequality (6.38) yields

$$\begin{aligned} h(V) &\leq \max\{n^2 r^2, nr^3, n^3 r\} \left( 28 + 400 \frac{\kappa^{d+1}}{\Delta_0} \deg(Z) + 5 \deg(Z) \right) \beta(Z) k_0 \\ &\leq 2^9 \max\{n^2 r^2, nr^3, n^3 r\} \max \left\{ 1, \frac{\kappa^{d+1}}{\Delta_0} \right\} \beta(Z) \deg(Z) k_0. \quad \square \end{aligned}$$

## 7. DEGREE AND HEIGHT UPPER BOUNDS FOR COMPACTIFICATIONS

Throughout this subsection let  $X \subsetneq \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $1 \leq r \leq n-1$ . Moreover, let  $\varphi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  be a homomorphism.

Using the open immersion  $\mathbf{G}_m^n \hookrightarrow \mathbf{P}^n$  introduced in Section 3.1 we consider the algebraic torus as an open subvariety of projective space. We let  $\overline{X}$  denote the Zariski closure of  $X$  in  $\mathbf{P}^n$ . Similarly we may consider  $\mathbf{G}_m^n \times \mathbf{G}_m^r$  as an open subvariety of  $\mathbf{P}^n \times \mathbf{P}^r$ . We let  $\overline{X}^\varphi$  denote the Zariski closure in  $\mathbf{P}^n \times \mathbf{P}^r$  of the graph of  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$ . Then  $\overline{X}^\varphi$  is an irreducible closed subvariety of  $\mathbf{P}^n \times \mathbf{P}^r$  of dimension  $r$ . There are open immersions  $X \rightarrow \overline{X}$  and  $X \rightarrow \overline{X}^\varphi$ . The same compactification was used in the earlier paper [22].

As in previous sections  $\pi_1$  and  $\pi_2$  denote the projections  $\mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^n$  and  $\mathbf{P}^n \times \mathbf{P}^r \rightarrow \mathbf{P}^r$ , respectively. We end up with a commutative diagram

$$(7.1) \quad \begin{array}{ccccc} \overline{X} & \longleftarrow & X & \xrightarrow{\varphi|_X} & \mathbf{G}_m^r \\ \parallel & & \downarrow & & \downarrow \\ \overline{X} & \longleftarrow & \overline{X}^\varphi & \xrightarrow{\pi_2|_{\overline{X}^\varphi}} & \mathbf{P}^r \\ & \pi_1|_{\overline{X}^\varphi} & & & \end{array}$$

of morphisms.

The purpose of this section is to bound degree and height of  $\overline{X}^\varphi$  in terms of  $X$  and  $\varphi$ . Recall that the degree of  $\overline{X}^\varphi$  is by definition the degree of  $s(\overline{X}^\varphi) \subset \mathbf{P}^{nr+n+r}$  where  $s$  is the Segre morphism. Recall that the height of  $\overline{X}^\varphi$  is the height of  $s(\overline{X}^\varphi)$  as a subvariety of  $\mathbf{P}^{nr+n+r}$ . Moreover,  $\Delta_i(\overline{X}^\varphi)$  are the bidegrees introduced in Section 6.

We use  $|\cdot|_\infty$  to denote the sup-norm of any matrix with real coefficients.

Homomorphisms  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  can be identified with  $n \times r$  matrices in integer coefficients. Therefore,  $|\varphi|_\infty$  is well-defined. It is non-zero if and only if  $\varphi$  is non-constant.

**Lemma 7.1.** *If  $\varphi$  is non-constant, then*

$$(7.2) \quad \deg(\overline{X}^\varphi) \leq (4n|\varphi|_\infty)^r \deg(X)$$

and

$$(7.3) \quad \Delta_i(\overline{X}^\varphi) \leq (4n)^r |\varphi|_\infty^{r-i} \deg(X) \quad \text{for all } 0 \leq i \leq r$$

and

$$(7.4) \quad \Delta_0(\overline{X}^\varphi) = \deg(\varphi|_X) \leq (4n|\varphi|_\infty)^r \deg(X).$$

*Proof.* We essentially follow the argument given in Lemma 3.3 [22]. The  $l^2$ -norm on  $\text{Mat}_{n,r}(\mathbf{R})$  was used in the reference, but here we work with the sup-norm. There it is shown that  $\overline{X}^\varphi$  is an irreducible component of  $(\overline{X} \times \mathbf{P}^r) \cap Y$  where  $Y$  is the set of common zeros of bihomogeneous polynomials whose bidegrees are at most  $(D, 1)$  with  $D \leq n|\varphi|_\infty$ .

By Philippon's version of Bézout's Theorem, Proposition 3.3 [29], we bound

$$(7.5) \quad \begin{aligned} H(D, 1; \overline{X}^\varphi) &\leq H(D, 1; \overline{X} \times \mathbf{P}^r) \\ &= \sum_{i=0}^{2r} \binom{2r}{i} (\pi_1^* \mathcal{O}(1)^i \pi_2^* \mathcal{O}(1)^{2r-i} [\overline{X} \times \mathbf{P}^r]) D^i. \end{aligned}$$

Commutativity of intersection products and the projection formula imply

$$(\pi_1^* \mathcal{O}(1)^i \pi_2^* \mathcal{O}(1)^{2r-i} [\overline{X} \times \mathbf{P}^r]) = \left( \mathcal{O}(1)^{2r-i} \pi_{2*} (\pi_1^* \mathcal{O}(1)^i [\overline{X} \times \mathbf{P}^r]) \right).$$

If  $i > r = \dim \overline{X}$ , then the intersection number on the right vanishes. On the other hand, if  $2r - i > r$  then it vanishes too because

$$(\pi_1^* \mathcal{O}(1)^i \pi_2^* \mathcal{O}(1)^{2r-i} [\overline{X} \times \mathbf{P}^r]) = \left( \mathcal{O}(1)^i \pi_{1*} (\pi_2^* \mathcal{O}(1)^{2r-i} [\overline{X} \times \mathbf{P}^r]) \right).$$

So only the term  $i = r$  survives in (7.5) and we obtain

$$H(D, 1; \overline{X}^\varphi) \leq \binom{2r}{r} (\pi_1^* \mathcal{O}(1)^r \pi_2^* \mathcal{O}(1)^r [\overline{X} \times \mathbf{P}^r]) D^r.$$

The intersection number on the right is  $\deg(X)$  and we conclude

$$H(D, 1; \overline{X}^\varphi) \leq \binom{2r}{r} \deg(X) D^r.$$

Inserting the definition of  $\Delta_i(\overline{X}^\varphi)$  from Section 6 leads us to

$$\sum_{i=0}^r \binom{r}{i} \Delta_i(\overline{X}^\varphi) D^i = H(D, 1; \overline{X}^\varphi) \leq \binom{2r}{r} \deg(X) D^r \leq 4^r \deg(X) D^r.$$

We note that the  $\Delta_i(\overline{X}^\varphi)$  cannot be negative. By Lemma 4.1 we have  $\deg(\overline{X}^\varphi) = \sum_{i=0}^r \binom{r}{i} \Delta_i(\overline{X}^\varphi)$ . So (7.2) and (7.3) follow from  $1 \leq D \leq n|\varphi|_\infty$ .

We have  $\Delta_0(\overline{X}^\varphi) = (\pi_2^* \mathcal{O}(1)^r [\overline{X}^\varphi])$  by definition. The projection formula implies

$$\Delta_0(\overline{X}^\varphi) = (\mathcal{O}(1)^r \pi_{2*} [\overline{X}^\varphi]) = \deg(\pi_2|_{\overline{X}^\varphi}) (\mathcal{O}(1)^r [\pi_2(\overline{X}^\varphi)]).$$

We have  $\deg(\pi_2|_{\overline{X}^\varphi}) = \deg(\varphi|_X)$  since the all vertical arrows in (7.1) are birational morphisms. Equality (7.4) certainly holds if  $\deg(\varphi|_X) = 0$ . Otherwise,  $\pi_2|_{\overline{X}^\varphi} : \overline{X}^\varphi \rightarrow \mathbf{P}^r$  has generically finite fibers and we have  $\dim \pi_2(\overline{X}^\varphi) = \dim \overline{X}^\varphi = r$  by the Fiber Dimension Theorem. Hence  $\pi_2(\overline{X}^\varphi) = \mathbf{P}^r$  and so  $\Delta_0(\overline{X}^\varphi) = \deg(\pi_2|_{\overline{X}^\varphi})$ , as desired.  $\square$

We need information on  $\kappa(\overline{X}^\varphi)$ , as defined in (6.2). Recall that we only defined this quantity if  $\Delta_0(\overline{X}^\varphi) > 0$  and  $\Delta_i(\overline{X}^\varphi) > 0$  for some  $1 \leq i \leq r$ .

**Lemma 7.2.** *We assume  $\deg(\varphi|_X) \geq 1$ . Then  $\Delta_0(\overline{X}^\varphi) > 0, \Delta_r(\overline{X}^\varphi) > 0$ , and we have*

$$\frac{|\varphi|_\infty}{(4n)^r \deg(X)} \frac{\deg(\varphi|_X)}{|\varphi|_\infty^r} \leq \kappa(\overline{X}^\varphi) \leq \deg(\varphi|_X).$$

*Proof.* Positivity of  $\Delta_0(\overline{X}^\varphi)$  follows from Lemma 7.1. By the projection formula we have  $\Delta_r(\overline{X}^\varphi) = \deg(\pi_1|_{\overline{X}^\varphi})(\mathcal{O}(1)^r[\pi_1(\overline{X}^\varphi)])$ . Recall that  $\overline{X}^\varphi \subset \mathbf{P}^n \times \mathbf{P}^r$  is the Zariski closure of the graph  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$ . This implies  $\deg(\pi_1|_{\overline{X}^\varphi}) = 1$  and  $\pi_1(\overline{X}^\varphi) = \overline{X}$ . So  $\Delta_r(\overline{X}^\varphi) = \deg(\overline{X}) > 0$ .

We use Lemma 7.1 to bound  $\kappa = \kappa(\overline{X}^\varphi)$  from above and below. Indeed, suppose  $1 \leq i \leq r$  with  $\Delta_i(\overline{X}^\varphi) > 0$  and  $\kappa = (\Delta_0(\overline{X}^\varphi)/\Delta_i(\overline{X}^\varphi))^{1/i}$ .

Then  $\kappa \leq \Delta_0(\overline{X}^\varphi)^{1/i} \leq \Delta_0(\overline{X}^\varphi)$  since  $\Delta_0(\overline{X}^\varphi) \geq 1$  and  $\Delta_i(\overline{X}^\varphi) \geq 1$ . The desired upper bound for  $\kappa$  follows from (7.4).

For the lower bound, (7.3) and (7.4) give

$$\kappa = \left( \frac{\Delta_0(\overline{X}^\varphi)}{\Delta_i(\overline{X}^\varphi)} \right)^{1/i} \geq |\varphi|_\infty \left( \frac{\deg(\varphi|_X)}{(4n)^r |\varphi|_\infty^r \deg(X)} \right)^{1/i} \geq |\varphi|_\infty \frac{\deg(\varphi|_X)}{(4n)^r |\varphi|_\infty^r \deg(X)},$$

in the second inequality we used (7.4).  $\square$

In Section 3.2 we introduced a height function  $h_s : \mathbf{G}_m^n(\overline{\mathbf{Q}}) \rightarrow [0, \infty)$  with sup-norm at the infinite places.

We now bound the height of  $\overline{X}^\varphi$ .

**Lemma 7.3.** *If  $\varphi \neq 0$ , then  $h(\overline{X}^\varphi) \leq (4n)^{n+2} |\varphi|_\infty^{r+1} \left( h(X) + \frac{\deg(X)}{|\varphi|_\infty} \right)$ .*

*Proof.* For any  $p \in \mathbf{G}_m^n(\overline{\mathbf{Q}})$  we have  $h_s(\varphi(p)) \leq nr|\varphi|_\infty h_s(p)$  by the discussion around (3.2). Therefore,  $h(\varphi(p)) \leq \frac{1}{2} \log(n+1) + nr|\varphi|_\infty h_s(p) \leq n + nr|\varphi|_\infty h(p)$  from the comparison estimates between the two heights. Let  $\epsilon > 0$ . We may find a Zariski dense set of  $p \in X(\overline{\mathbf{Q}})$  with  $h(p) \leq \mu^{\text{ess}}(X) + \epsilon$ . Recall that  $s$  is the Segre morphism. We have  $h(s(p, \varphi(p))) = h(p) + h(\varphi(p)) \geq h(p)$ , so

$$h(s(p, \varphi(p))) \leq \mu^{\text{ess}}(X) + \epsilon + n + nr|\varphi|_\infty h(p) \leq n + (1 + nr|\varphi|_\infty)(\mu^{\text{ess}}(X) + \epsilon).$$

The resulting set of points  $(p, \varphi(p))$  lies Zariski dense in  $\overline{X}^\varphi$ . So

$$\mu^{\text{ess}}(s(\overline{X}^\varphi)) \leq n + 2nr|\varphi|_\infty(\mu^{\text{ess}}(X) + \epsilon)$$

and letting  $\epsilon$  tend to zero gives

$$(7.6) \quad \mu^{\text{ess}}(s(\overline{X}^\varphi)) \leq n + 2nr|\varphi|_\infty \mu^{\text{ess}}(X).$$

Zhang's inequality (6.18) lets us compare essential minimum with the height and degree of a variety. More precisely, we have

$$\mu^{\text{ess}}(s(\overline{X}^\varphi)) \geq \frac{h(\overline{X}^\varphi)}{(1+r)\deg(\overline{X}^\varphi)} \quad \text{and} \quad \mu^{\text{ess}}(X) \leq \frac{h(X)}{\deg(X)}.$$



Combining these two with (7.6) gives

$$\frac{h(\overline{X}^\varphi)}{(1+r)\deg(\overline{X}^\varphi)} \leq n + 2nr|\varphi|_\infty \frac{h(X)}{\deg(X)}$$

and so

$$h(\overline{X}^\varphi) \leq n^2 \deg(\overline{X}^\varphi) + 2n^3 |\varphi|_\infty \frac{\deg(\overline{X}^\varphi)}{\deg(X)} h(X)$$

after using  $r \leq n-1$ . Lemma 7.1 implies  $\deg(\overline{X}^\varphi) \leq (4n|\varphi|_\infty)^r \deg(X)$  and this completes the proof.  $\square$

## 8. TROPICAL GEOMETRY AND DEGREE LOWER BOUNDS

Recall that  $n$  and  $r$  are positive integers. As usual, we identify elements of  $\text{Mat}_{r,n}(\mathbf{Z})$  with homomorphisms of algebraic groups  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$ . If  $\varphi$  is such a matrix and  $X \subset \mathbf{G}_m^n$  is an irreducible closed subvariety defined over  $\mathbf{C}$  of dimension  $r$ , then  $\deg(\varphi|_X)$  is the degree of the restriction  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$ . In this section we use techniques from Tropical Geometry to evaluate  $\deg(\varphi|_X)$  in terms of  $\varphi$  and the so-called tropicalization of  $X$ .

Although a  $\varphi \in \text{Mat}_{r,n}(\mathbf{Q})$  need not determine a homomorphism of algebraic groups we can make sense of  $\deg(\varphi|_X)$  by killing denominators, cf. Lemma 3.1(iii) [22]. In this reference the author showed

$$\deg(\lambda\varphi|_X) = |\lambda|^r \deg(\varphi|_X)$$

for all  $\varphi \in \text{Mat}_{r,n}(\mathbf{Q})$  and  $\lambda \in \mathbf{Q}$ .

Let  $s$  be an integer with  $r \leq s \leq n$ . Let  $\epsilon > 0$ , then  $\varphi_0 \in \text{Mat}_{s,n}(\mathbf{R})$  is called  $\epsilon$ -regular if for any  $\varphi \in \text{Mat}_{s,n}(\mathbf{R})$  with  $\text{rank}(\varphi) < s$  we have  $|\varphi_0 - \varphi|_\infty \geq \epsilon$ . In other words, the distance of  $\varphi_0$  to the set of matrices of non-full rank is at least  $\epsilon$ .

Let  $\Pi_{rs}$  denote the set of  $r \times s$  matrices which represent projections  $\mathbf{Q}^s \rightarrow \mathbf{Q}^r$  onto  $r$  distinct coordinates of  $\mathbf{Q}^s$ .

**Proposition 8.1.** *Let  $X \subsetneq \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\mathbf{C}$  with  $\dim X = r \geq 1$ . Let  $s$  be an integer with  $r \leq s \leq n$ . One of the following two cases holds.*

- (i) *There exists an algebraic subgroup  $H \subset \mathbf{G}_m^n$  such that  $\dim_p X \cap pH \geq \max\{1, s + \dim H - n + 1\}$  for all  $p \in X(\mathbf{C})$ .*
- (ii) *If  $\epsilon \in (0, 1]$  and if  $\varphi \in \text{Mat}_{s,n}(\mathbf{Q})$  is  $\epsilon$ -regular, there is  $\pi \in \Pi_{rs}$  with*

$$\deg(\pi\varphi|_X) \geq 2^{-50n^5(2n)^{n^2}} \deg(X)^{-\frac{1}{2}(n-r)(r+1+(r+3)(sn+1)(2r)^{sn})} \left( \frac{\epsilon}{\max\{1, |\varphi|_\infty\}} \right)^{sn(2r)^{sn}}.$$

**8.1. Preliminaries on Tropical Geometry.** Let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\mathbf{C}$  of dimension  $r \geq 1$ . One can associate to  $X$  a set  $\mathcal{T}(X) \subset \mathbf{Q}^n$ , called the tropicalization of  $X$ , in the follow manner.

Let  $K$  be the field of puiseux series with complex coefficients. This is an algebraically closed field equipped with a surjective valuation  $\text{ord} : K \rightarrow \mathbf{Q} \cup \{+\infty\}$ . We also use  $\text{ord} : K^n \rightarrow (\mathbf{Q} \cup \{+\infty\})^n$  to denote the  $n$ -fold product. Let  $X_K$  denote  $X$  considered as a variety over  $K$ . We set

$$(8.1) \quad \mathcal{T}(X) = \{\text{ord}(x); x \in X_K(K)\} \subset \mathbf{Q}^n.$$

The closure of  $\mathcal{T}(X)$  in  $\mathbf{R}^n$  coincides with the so-called Bieri-Groves set of  $X$ , see the work of Einsiedler, Kapranov, and Lind [16] for a proof. The Bieri-Groves set of  $X$  is known [3] to be a rational polyhedral set of pure dimension  $r$ . It follows from the argument given by Einsiedler, Kapranov, and Lind that  $\mathcal{T}(X)$  is the intersection of a rational polyhedral set with  $\mathbf{Q}^n$ .

In our particular situation, the variety  $X$  is defined over  $\mathbf{C}$ . Because  $\text{ord}$  is trivial on  $\mathbf{C}$  we find that  $\mathcal{T}(X)$  is a finite union of rational polyhedral cones of pure dimension  $r$ . In other words,  $\mathcal{T}(X)$  is a finite union of sets

$$\{(x_1, \dots, x_n) \in \mathbf{Q}^n; a_{i1}x_1 + \dots + a_{in}x_n \geq 0 \text{ for } 1 \leq i \leq n\}$$

with  $a_{ij} \in \mathbf{Z}$ . Pure dimension  $r$  means that the vector subspace of  $\mathbf{Q}^n$  spanned by each of the sets has dimension  $r$ .

We call  $v \in \mathcal{T}(X)$  regular if some neighborhood  $v$  in  $\mathcal{T}(X)$  coincides with the neighborhood of a vector subspace of  $\mathbf{Q}^n$ . In this case the vector subspace is uniquely determined and we denote it by  $L_v$ . We define  $\mathcal{T}^0(X) \subset \mathcal{T}(X)$  to be the set of regular points. It follows that  $\dim L_v = \dim X$  for  $v \in \mathcal{T}^0(X)$ . We define the finite set

$$\Sigma(X) = \{L_v; v \in \mathcal{T}^0(X)\}.$$

There is more information attached to the tropicalization of  $X$ . One can define a locally constant function  $m_X : \mathcal{T}^0(X) \rightarrow \mathbf{N}$  called the multiplicity function, cf. Definition 3.7 [44].

*Remark 8.1.* We have  $\mathcal{T}(\mathbf{G}_m^n) = \mathbf{Q}^n$ , this is clear from our characterization (8.1). Hence  $\mathcal{T}^0(\mathbf{G}_m^n) = \mathbf{Q}^n$  too. If  $v \in \mathbf{Q}^n$ , then  $m_{\mathbf{G}_m^n}(v) = 1$  by Corollary 3.15 [44].

Our main tool from tropical geometry is a special case of a result of Sturmfels and Tevelev [44]. It allows us to determine  $\deg(\varphi|_X)$  for varying  $\varphi$  (and fixed  $X$ ).

In the formulation below we regard  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$  simultaneously as a homomorphism  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$  and a linear map  $\mathbf{Q}^n \rightarrow \mathbf{Q}^r$ .

If we for the moment drop the assumption that  $X$  has dimension  $r$ , then

$$(8.2) \quad \mathcal{T}(\overline{\varphi(X)}) = \varphi(\mathcal{T}(X))$$

holds by Remark 2.1 [44].

**Theorem 12** (Sturmfels, Tevelev). *Let  $X, n, \Sigma(X) = \{L_v\}$ , and  $r$  be as above and let  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$  such that the restriction  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$  is dominant with degree  $\deg(\varphi|_X)$ . Then  $\varphi|_{\mathcal{T}(X)} : \mathcal{T}(X) \rightarrow \mathbf{Q}^r$  is surjective. Moreover, assume  $w \in \mathbf{Q}^r$  such that  $\varphi|_{\mathcal{T}(X)}^{-1}(w)$  is a finite subset of  $\mathcal{T}^0(X)$ , then*

$$(8.3) \quad \deg(\varphi|_X) = \sum_{v \in \varphi|_{\mathcal{T}(X)}^{-1}(w)} m_X(v) [\mathbf{Z}^r : \varphi(L_v \cap \mathbf{Z}^n)].$$

*Proof.* This follows from Theorem 3.12 [44]. We remark that  $\varphi|_X : X \rightarrow \mathbf{G}_m^r$  is generically finite since it is dominant and  $\dim X = r$ . Moreover,  $m_{\mathbf{G}_m^r}(w) = 1$  for all  $w \in \mathbf{Q}^r$  by Remark 8.1.  $\square$

*Remark 8.2.* Let  $v$  be as in the sum (8.3) and  $v' \in L_v \cap \mathbf{Z}^n$  with  $\varphi(v') = 0$ . Then  $v + \lambda v' \in \mathcal{T}^0(X)$  for  $\lambda \in \mathbf{Q}$  sufficiently small. Since  $\varphi|_{\mathcal{T}(X)}^{-1}(w)$  is finite we must have  $v' = 0$ . Hence

$\varphi|_{L_v \cap \mathbf{Z}^n}$  is injective and therefore  $\varphi(L_v \cap \mathbf{Z}^n)$  has rank equal to  $\dim L_v = \dim X = r$ . In particular,  $\varphi(L_v \cap \mathbf{Z}^n)$  has finite index in  $\mathbf{Z}^r$ , and the sum above is well-defined.

A similar argument shows that if the sum in (8.3) contains terms corresponding to two different  $v$  and  $v'$ , then  $L_v \neq L_{v'}$ .

In order to get explicit estimates from this theorem we need to get a grip on  $\Sigma(X)$  and the multiplicity function for a given  $X$ .

We recall that  $\deg(X)$  is the degree of the Zariski closure of  $X$  in  $\mathbf{P}^n$ .

**Lemma 8.1.** *Let  $v \in \mathcal{T}^0(X)$ , then  $m_X(v) \leq \deg(X)$ .*

*Proof.* Since  $m_X$  is locally constant on  $\mathcal{T}^0(X)$  it suffices to prove the inequality for some  $v' \in \mathcal{T}^0(X)$  sufficiently close to  $v$ . We note also that  $L_v = L_{v'}$  has dimension  $r$ . Let  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$  be a projection onto  $r$  distinct coordinates with  $\varphi|_{L_v}$  bijective.

Let  $Y$  be the Zariski closure of  $\varphi(X)$  in  $\mathbf{G}_m^r$ . By (8.2) the set  $\mathcal{T}(Y)$  contains the image of a neighborhood of  $v$  in  $L_v$ . Hence  $\dim Y = \dim \mathcal{T}(Y) = r$  and we conclude that  $Y = \mathbf{G}_m^r$ . Therefore,  $\varphi|_X$  is dominant.

We may apply Theorem 12 to  $\varphi$  and an appropriate  $w \in \mathbf{Q}^r$  which we proceed to choose. The fiber  $\varphi|_{L_v}^{-1}(w)$  is finite by our choice of  $\varphi$  regardless of  $w$ . We recall that  $\mathcal{T}(X)$  is contained in a finite union of vector subspaces of  $\mathbf{Q}^n$ . Hence, after replacing  $v$  by a sufficiently close  $v' \in L_v \cap \mathcal{T}^0(X)$  we may assume that  $\varphi|_{\mathcal{T}(X)}^{-1}(w)$  satisfies the necessary conditions. Now (8.3) implies  $\deg(\varphi|_X) \geq m_X(v)[\mathbf{Z}^r : \varphi(L_v \cap \mathbf{Z}^n)] \geq m_X(v)$ . An application of Bézout's Theorem leads to  $\deg(\varphi|_X) \leq \deg(X)$  and the lemma follows.  $\square$

**Lemma 8.2.** *We have  $\#\Sigma(X) \leq 2^{5n^3} \deg(X)^{(r+1)(n-r)}$  and if  $L \in \Sigma(X)$  there is  $\psi \in \text{Mat}_{n-r,n}(\mathbf{Z})$  with  $L = \ker \psi \subset \mathbf{Q}^n$  and  $|\psi|_\infty \leq 2n \deg(X)$ .*

*Proof.* We may assume  $r = \dim X \leq n-1$ , otherwise  $X = \mathbf{G}_m^n$  and then  $\Sigma(X)$  contains only  $\mathbf{Q}^n$ .

For an index set  $I = \{i_1 < \dots < i_{r+1}\} \subset \{1, \dots, n\}$  we let  $\varphi_I : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^{r+1}$  denote the projection onto the coordinates  $i_1, \dots, i_{r+1}$ . The Zariski closure of  $\varphi_I(X)$  in  $\mathbf{G}_m^{r+1}$  has dimension at most  $r$ . Moreover, its degree is at most  $\deg(X)$ . Hence it is in the zero set of a non-constant polynomial  $f_I$  in  $r+1$  variables and degree at most  $n \deg(X)$ ; for the latter statement we refer to Chardin's Corollaire 2, cf. Exemple 1 [12].

Say  $v \in \mathcal{T}(X)$ . So there is  $x \in X_K(K)$  with  $v = \text{ord}(x)$ .

We have  $f_I(\varphi_I(x)) = 0$ . Because the coefficients of  $f_I$  have valuation zero we must have  $\text{ord}(x^{v_I}) = 0$  with  $v_I \in \mathbf{Z}^{r+1} \setminus \{0\}$  the difference of two distinct elements in the support of  $f_I$ . On letting  $I$  vary we find  $n-r$  independent  $v_I \in \mathbf{Z}^n$  with  $\text{ord}(x^{v_I}) = 0$ . These define the rows of  $\psi \in \text{Mat}_{\mathbf{Z}}(n-r, n)$  with rank  $r$  whose kernel contains  $\text{ord}(x)$ . In total there are  $\binom{n}{r+1}$  possibilities for  $I$  and there are  $\binom{\binom{n}{r+1}}{n-r} \leq \binom{n}{r+1}^{n-r} \leq 2^{n(n-r)}$  possibilities to choose  $n-r$  different  $I$ . The sup-norm of the difference of two elements in the support of  $f_I$  is bounded by  $2 \deg(f_I) \leq 2n \deg(X)$ . This leaves us with at most  $(1 + 4n \deg(X))^{r+1}$  possibilities for  $v_I$ . In total we obtain at most  $2^{n(n-r)}(1 + 4n \deg(X))^{(r+1)(n-r)}$  different

$\psi$ . Using  $r \geq 1$  and  $r \leq n-1$  we estimate

$$\begin{aligned} 2^{n(n-r)}(1+4n\deg(X))^{(r+1)(n-r)} &\leq 2^{n(n-r)}(8n\deg(X))^{(r+1)(n-r)} \\ &\leq 2^{4n(n-r)}n^{(r+1)(n-r)}\deg(X)^{(r+1)(n-r)} \\ &\leq 2^{4n(n-r)+n(r+1)(n-r)}\deg(X)^{(r+1)(n-r)} \\ &\leq 2^{5n^3}\deg(X)^{(r+1)(n-r)}. \end{aligned}$$

Since  $v \in \mathcal{T}(X)$  was arbitrary we conclude that  $\mathcal{T}(X)$  is contained in a union of at most  $2^{5n^3}\deg(X)^{(r+1)(n-r)}$  vector subspaces of  $\mathbf{Q}^n$  determined by a  $\psi$  as above. The lemma follows since each  $L \in \Sigma(X)$  has the same dimension as the kernel of a  $\psi$ .  $\square$

If  $L \in \Sigma(X)$ , then  $L \cap \mathbf{Z}^n \subset \mathbf{Z}^n$  is a subgroup of rank  $r$ . In the next lemma we find a lattice basis for  $L \cap \mathbf{Z}^n$  with controlled entries.

**Lemma 8.3.** *For  $L \in \Sigma(X)$  there is  $B_L \in \text{Mat}_{n,r}(\mathbf{Z})$  whose columns  $v_1, \dots, v_r$  are a basis for  $L \cap \mathbf{Z}^n$  with  $|v_1|_\infty \cdots |v_r|_\infty \leq 2^n r! n^{2n} \deg(X)^{n-r}$ .*

*Proof.* Lemma 8.2 supplies us with  $\psi \in \text{Mat}_{n-r,n}(\mathbf{Z})$  of rank  $r$  such that  $\ker \psi = L \subset \mathbf{Q}^n$  and  $|\psi|_\infty \leq 2n\deg(X)$ . By Corollary 2.9.9 [3] there are independent  $v'_1, \dots, v'_r \in \mathbf{Z}^n$  with  $\psi(v'_1) = \cdots = \psi(v'_r) = 0$  and  $\prod_{k=1}^r |v'_k|_\infty \leq n^{(n-r)/2} |\psi|_\infty^{n-r}$ . We note that our reference works with the multiplicative projective height. Since our coefficients are integers, we may compare this height to the norm  $|\cdot|_\infty$ . This is possible since we suppose, as we may, that the entries of each  $v'_k$  are coprime. After permuting the  $v'_k$  we may suppose  $|v'_1|_\infty \leq \cdots \leq |v'_r|_\infty$ .

By a result of Mahler there is a basis  $(v_1, \dots, v_k)$  of  $\ker \psi$  with  $|v_k|_\infty \leq k|v'_k|_\infty$ . This statement is similar to Lemma 3.2.11 [4]; said reference works with a different norm and gives a somewhat different statement. But its proof adapts easily to yield the statement above.

We conclude  $\prod_{k=1}^r |v_k|_\infty \leq r! n^{(n-r)/2} |\psi|_\infty^{n-r} \leq 2^n r! n^{2n} \deg(X)^{n-r}$  as desired.  $\square$

In the following proposition we use  $B_L$  as given by the previous lemma.

**Proposition 8.2.** *Let  $\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$ .*

(i) *We have*

$$(8.4) \quad \deg(\varphi|_X) \leq \deg(X) \max_{L \in \Sigma(X)} \{|\det \varphi B_L|\} \# \Sigma(X).$$

(ii) *If  $L \in \Sigma(X)$  then*

$$\deg(\varphi|_X) \geq |\det \varphi B_L|.$$

*Proof.* In the proof of the proposition we will use

$$[\mathbf{Z}^r : \varphi(L \cap \mathbf{Z}^n)] = |\det \varphi B_L|$$

for  $L \in \Sigma(X)$  where we define the index to be 0 if  $\varphi(L \cap \mathbf{Z}^n)$  has rank less than  $r$ .

We begin with part (i). It suffices to assume that  $\varphi|_X$  is dominant. We note that  $m_X(v) \leq \deg(X)$  for all  $v \in \mathcal{T}^0(X)$  by Lemma 8.1. Inequality (8.4) follows from Sturmfels and Tevelev's result if there exists  $w \in \mathbf{Q}^r$  such that  $\varphi|_{\mathcal{T}^0(X)}^{-1}(w)$  is a finite subset of  $\mathcal{T}^0(X)$ . Indeed, for  $w$  outside a finite union of proper vector subspace of  $\mathbf{Q}^r$

the fiber  $\varphi|_{\mathcal{T}(X)}^{-1}(w)$  does not meet any  $L' \in \Sigma(X)$  with  $\varphi|_{L'}$  non-injective or any points of  $\mathcal{T}(X) \setminus \mathcal{T}^0(X)$

Let us now prove part (ii). Without loss of generality we may assume  $\det \varphi B_L \neq 0$ . In other words,  $\varphi|_L : L \rightarrow \mathbf{Q}^r$  is bijective.

We remark that the tropicalization of the Zariski closure of  $\varphi(X)$  is all  $\mathbf{Q}^r$  by (8.2). In particular,  $\varphi|_X$  is dominant.

In view of the Sturmfels and Tevelev's result, it suffices to show that there is  $w \in \mathbf{Q}^r$  such that  $\varphi|_{\mathcal{T}(X)}^{-1}(w)$  is a finite subset of  $\mathcal{T}^0(X)$ . This is the case by the same argument we gave in the proof of part (i).  $\square$

**8.2. Ax's Theorem and Degree Lower Bounds.** The following proposition is a variant of the author's Proposition 7.3 [22]. Its proof relies on Ax's Theorem [1].

**Proposition 8.3.** *Let  $X$  be an irreducible closed subvariety of  $\mathbf{G}_m^n$  defined over  $\mathbf{C}$  of dimension  $1 \leq r \leq n - 1$ . Let  $s$  be an integer with  $r \leq s \leq n$ . Assume  $\varphi_0 \in \text{Mat}_{s,n}(\mathbf{R})$  has rank  $s$ . Then one of the following two cases holds.*

- (i) *There exists an algebraic subgroup  $H \subset \mathbf{G}_m^n$  such that  $\dim_p X \cap pH \geq \max\{1, s + \dim H - n + 1\}$  for all  $p \in X(\mathbf{C})$ .*
- (ii) *There exist  $\epsilon > 0$  and an open neighborhood  $U$  of  $\varphi_0$  in  $\text{Mat}_{s,n}(\mathbf{R})$  such that  $\max_{\pi \in \Pi_{rs}} \deg(\pi\varphi|_X) \geq \epsilon$  for all  $\varphi \in U \cap \text{Mat}_{s,n}(\mathbf{Q})$ .*

*Proof.* Take  $\mathcal{K} = \{\varphi_0\}$  in Proposition 7.3 [22].  $\square$

Using the results from tropical geometry in the previous subsection we will eventually transform this qualitative lower bound into a quantitative one. At first we show a non-vanishing result.

Throughout this subsection, and if not stated otherwise, we keep the notation of Proposition 8.3.

**Lemma 8.4.** *Let us assume part (i) of Proposition 8.3 does not hold for  $X$ . If  $\varphi_0 \in \text{Mat}_{s,n}(\mathbf{R})$  has rank  $s$ , then there exist  $L \in \Sigma(X)$  and  $\pi \in \Pi_{rs}$  with*

$$\det(\pi\varphi_0 B_L) \neq 0.$$

*Proof.* Let  $U \subset \text{Mat}_{s,n}(\mathbf{R})$  be the neighborhood of  $\varphi_0$  from Proposition 8.3. Let  $\varphi \in U \cap \text{Mat}_{s,n}(\mathbf{Q})$  and  $\lambda \in \mathbf{N}$  with  $\lambda\varphi \in \text{Mat}_{s,n}(\mathbf{Z})$ . We have  $\max_{\pi \in \Pi_{rs}} \deg(\lambda\pi\varphi|_X) = \lambda^r \max_{\pi \in \Pi_{rs}} \deg(\pi\varphi|_X) \geq \lambda^r \epsilon$ . On the other hand, Proposition 8.2(i) implies

$$\begin{aligned} \deg(\lambda\pi\varphi|_X) &\leq \deg(X) \max_{L \in \Sigma(X)} |\det(\lambda\pi\varphi B_L)| \# \Sigma(X) \\ &= \deg(X) \lambda^r \max_{L \in \Sigma(X)} |\det(\pi\varphi B_L)| \# \Sigma(X) \end{aligned}$$

for all  $\pi \in \Pi_{rs}$ . We cancel  $\lambda$  and conclude

$$\max_{\pi \in \Pi_{rs}} \max_{L \in \Sigma(X)} |\det(\pi\varphi B_L)| \geq \epsilon > 0$$

for all  $\varphi \in U \cap \text{Mat}_{s,n}(\mathbf{Q})$ .

By continuity and since  $U \cap \text{Mat}_{s,n}(\mathbf{Q})$  lies dense in  $U$  we have the same inequality for  $\varphi_0$ . In particular,  $\det(\pi\varphi_0 B_L) \neq 0$  for some  $L$  and  $\pi$ .  $\square$

The following corollary is an consequence of the lemma above. It will be of no further relevance for the current article. Although it would be interesting to know if it could be proved using methods from tropical geometry instead of Ax's Theorem.

We let  $\overline{\mathcal{T}(X)}$  be the closure of  $\mathcal{T}(X) \subset \mathbf{Q}^n$  in  $\mathbf{R}^n$ .

Let us now prove Corollary 1 which is stated in Section 2.

*Proof.* For  $\varphi_0$  as in the hypothesis, Lemma 8.4 implies that we are in alternative (i) of Proposition 8.3. Let  $H$  be the algebraic subgroup mentioned there. We fix a surjective homomorphism  $\varphi' : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^{n-\dim H}$  with kernel  $H$ .

Say  $n - \dim H \geq r$ . Then we compose  $\varphi'$  with the projection onto the first  $r$  coordinates of  $\mathbf{G}_m^{n-\dim H}$  and obtain a surjective homomorphism  $\varphi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$ . Each fiber of  $\varphi|_X$  has dimension at least  $\max\{1, s + \dim H - n + 1\}$ . By the Fiber Dimension Theorem, the Zariski closure of  $\varphi(X)$  has dimension at most  $\dim X - \max\{1, s + \dim H - n + 1\} < \dim X$ . If we consider  $\varphi$  as a matrix in  $\text{Mat}_{r,n}(\mathbf{Q})$ , then it has rank  $r$ . Moreover, by (8.2) we have  $\varphi(\mathcal{T}(X)) \neq \mathbf{Q}^r$ .

Now let us assume  $n - \dim H < r$ . We remark that

$$s + \dim H - n + 1 \geq r + \dim H - n + 1 > 1.$$

We take the product of  $\varphi'$  with some homomorphism  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^{r-(n-\dim H)}$  in general position to obtain a surjective homomorphism  $\varphi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$ . By intersection theory, the fibers of  $\varphi|_X$  have dimension at least

$$s + \dim H - n + 1 - (r - (n - \dim H)) = s - r + 1 \geq 1.$$

As before,  $\varphi(X)$  does not lie Zariski dense in  $\mathbf{G}_m^r$  and we also conclude  $\varphi(\mathcal{T}(X)) \neq \mathbf{Q}^r$ .  $\square$

**Example.** The previous corollary shows that the collection of  $\mathbf{Q}$ -vector spaces  $\{L_v\}$  associated to  $X$  satisfy a certain rationality condition. Indeed, it can be reformulated as follows. Let  $\overline{L_v}$  be the closure of  $L_v$  in  $\mathbf{R}^n$  or equivalently, the vector subspace of  $\mathbf{R}^n$  generated by  $L_v$ . If there exists a vector subspace  $V_0 \subset \mathbf{R}^n$  of dimension  $n - r$  with

$$\overline{L_v} \cap V_0 \neq 0 \quad \text{for all } v$$

then there exists a vector subspace  $V \subset \mathbf{Q}^n$  of dimension  $n - r$  with

$$L_v \cap V \neq 0 \quad \text{for all } v.$$

We exhibit four 2-dimensional vector subspaces in  $\mathbf{Q}^4$  that do not satisfy this rationality condition. The four bases

$$\begin{aligned} &((1, 0, 1, 0), (0, -2, 0, 1)), \\ &((1, -1, 0, 0), (0, 0, 1, 1)), \\ &((0, 1, 0, 0), (0, 0, 1, 0)), \quad \text{and} \\ &((1, 0, 0, 0), (0, 0, 0, 1)) \end{aligned}$$

determine four planes  $L_1, L_2, L_3$ , and  $L_4$  in  $\mathbf{Q}^4$ , respectively.

Say  $V \subset \mathbf{Q}^4$  is an arbitrary 2-dimensional vector subspace of  $\mathbf{Q}^4$ . One can verify that  $V \cap L_i = 0$  for some  $i \in \{1, 2, 3, 4\}$ .

On the other hand, each  $L_i$  meets the vector subspace of  $\mathbf{R}^4$  with basis

$$((0, \sqrt{2}, 1, 0), (-\sqrt{2}, 0, 0, 1))$$



non-trivially.

In particular,  $\Sigma(X) \neq \{L_1, L_2, L_3, L_4\}$  for all irreducible surfaces  $X \subset \mathbf{G}_m^4$ . Sam Payne has pointed out to the author that this also follows from the fact that the tropicalization of a variety is connected in codimension one. Indeed, the pairwise intersection of the  $L_i$  is trivial.

Let  $M_{ij}$  be independent variables for  $1 \leq i \leq s$  and  $1 \leq j \leq n$ . They are entries of an  $s \times n$  matrix  $M = (M_{ij})$ . We define the polynomial

$$D_X = \sum_{\pi \in \Pi_{rs}} \sum_{L \in \Sigma(X)} \det(\pi M B_L)^2 \in \mathbf{Z}[M_{ij}].$$

Lemma 8.4 tells us that if  $X$  is not as in alternative (i) of Proposition 8.3 then

$$(8.5) \quad \varphi_0 \in \text{Mat}_{s,n}(\mathbf{R}) \quad \text{with} \quad D_X(\varphi_0) = 0 \quad \text{implies} \quad \text{rank}(\varphi_0) < s.$$

This polynomial is closely related to the degree map  $\varphi \mapsto \deg(\varphi|_X)$ . It has the advantage that it can be evaluated for  $\varphi$  with real entries. Recall that  $\deg(\varphi|_X)$  has no straightforward interpretation for irrational  $\varphi$ .

Since we can contain the vanishing locus of  $D_X$  we will obtain an explicit lower bound for its values. To do this we must first determine basic properties of  $D_X$ .

For a polynomial  $P$  in any number of variables and integer coefficients we let  $|P|_\infty$  denote the largest absolute value of any coefficient.

**Lemma 8.5.** (i) *The polynomial  $D_X$  is homogeneous of degree  $2r$ .*

(ii) *We have  $|D_X|_\infty \leq 2^{20n^3} \deg(X)^{(r+3)(n-r)}$ .*

*Proof.* Part (i) follows from properties of the determinant.

We turn to Part (ii). First we remark that if  $P_1, \dots, P_k$  are polynomials in any number of variables, then  $|P_1 + \dots + P_k|_\infty \leq |P_1|_\infty + \dots + |P_k|_\infty$  by the triangle inequality. The product can be bounded using Lemma 1.6.11 [4] as  $|P_1 \cdots P_k|_\infty \leq 2^d |P_1|_\infty \cdots |P_k|_\infty$  where  $d$  is the sum of the partial degrees of  $P_1 \cdots P_k$ .

Say  $L \in \Sigma(X)$  and  $\pi \in \Pi_{rs}$ . Let  $v_1, \dots, v_r \in \mathbf{Z}^n$  be the columns of  $B_L$ ; latter is given by Lemma 8.3. An entry in the  $j$ -th column of  $\pi M B_L$  is a linear form with coefficients among the coefficients of  $v_j$ . Each term in the Leibniz formula for  $\det(\pi M B_L)$  is a polynomial in at most  $nr$  distinct  $M_{ij}$  with each partial degree at most 1. Since there are  $r!$  such terms we obtain  $|\det(\pi M B_L)|_\infty \leq 2^{nr} r! |v_1|_\infty \cdots |v_r|_\infty$ . Each partial degree of  $\det(\pi M B_L)^2$  is at most 2 and at most  $nr$  variables appear, so  $|\det(\pi M B_L)^2|_\infty \leq 2^{2nr} |\det(\pi M B_L)|_\infty^2 \leq 2^{4nr} r!^2 |v_1|_\infty^2 \cdots |v_r|_\infty^2$ . The bound from Lemma 8.3 leads to

$$(8.6) \quad |\det(\pi M B_L)^2|_\infty \leq 2^{4nr} r!^2 (2^n r! n^{2n} \deg(X)^{n-r})^2.$$

Now  $D_X$  consists of  $\#\Pi_{rs} \# \Sigma = \binom{s}{r} \# \Sigma \leq 2^s \# \Sigma$  terms for the form  $\det(\pi M B_L)^2$ . We use the bound for  $\# \Sigma \leq 2^{5n^3} \deg(X)^{(r+1)(n-r)}$  from Lemma 8.2 and (8.6) to deduce

$$\begin{aligned} |D_X|_\infty &\leq 2^s 2^{5n^3} \deg(X)^{(r+1)(n-r)} 2^{4nr} r!^2 (2^n r! n^{2n} \deg(X)^{n-r})^2 \\ &\leq 2^{s+5n^3+4nr+2n} n^{4n} r!^4 \deg(X)^{(r+1)(n-r)+2(n-r)} \\ &\leq 2^{12n^3} n^{8n} \deg(X)^{(r+3)(n-r)} \\ &\leq 2^{12n^3} 2^{8n^2} \deg(X)^{(r+3)(n-r)} \end{aligned}$$

where we used  $r, s \leq n-1$  and  $n \leq 2^n$ . □



We now prove Proposition 8.1.

Let us assume that we are not in alternative (i). Then we are not in alternative (i) of Proposition 8.3 and the conclusion of Lemma 8.4 holds.

By (8.5) we know that the set of zeros of  $D_X$  is contained in the set of matrices of  $\text{Mat}_{s,n}(\mathbf{R})$  with non-full rank.

Let  $\varphi$  and  $\epsilon$  be as in the hypothesis. We apply Rémond's explicit Łojasiewicz inequality [40] to bound  $D_X(\varphi)$  from below.

Recall that  $D_X$  is a polynomial of degree  $2r$  in  $sn$  variables  $M_{ij}$ . Rémond's inequality and Lemma 8.5(i) imply

$$D_X(\varphi) = |D_X(\varphi)| \geq (e^{4rsn} |D_X|_\infty)^{-(sn+1)(2r)^{sn}} \delta \geq 2^{-12rs^2n^2(2r)^{sn}} |D_X|_\infty^{-(sn+1)(2r)^{sn}} \delta$$

where for brevity we set

$$\delta = \left( \frac{\epsilon}{\max\{1, |\varphi|_\infty\}} \right)^{sn(2r)^{sn}} \leq 1.$$

Next we use the bound for  $|D_X|_\infty$  given by Lemma 8.5(ii); using  $n^n \leq 2^{n^2}$  we find

$$D_X(\varphi) \geq 2^{-12rs^2n^2(2r)^{sn} - 20n^3(sn+1)(2r)^{sn}} \deg(X)^{-(sn+1)(2r)^{sn}(r+3)(n-r)} \delta.$$

The exponent of  $2^{-1}$  is

$$12rs^2n^2(2r)^{sn} + 20n^3(sn+1)(2r)^{sn} \leq (12 + \frac{3}{2}20)n^5(2n)^{n^2} \leq 50n^5(2n)^{n^2}$$

where we used  $s, r \leq n$  and  $n \geq 2$ . We conclude

$$(8.7) \quad D_X(\varphi) \geq 2^{-50n^5(2n)^{n^2}} \deg(X)^{-(sn+1)(2r)^{sn}(r+3)(n-r)} \delta.$$

By definition of  $D_X$  there exist  $L \in \Sigma(X)$  and  $\pi \in \Pi_{rs}$  with

$$D_X(\varphi) \leq \#\Pi_{rs} \# \Sigma(X) \det(\pi\varphi B_L)^2.$$

Lemma 8.2 gives

$$D_X(\varphi) \leq 2^s 2^{5n^3} \deg(X)^{(r+1)(n-r)} \det(\pi\varphi B_L)^2 \leq 2^{10n^3} \deg(X)^{(r+1)(n-r)} \det(\pi\varphi B_L)^2.$$

Together with (8.7) we conclude

$$(8.8) \quad \det(\pi\varphi B_L)^2 \geq 2^{-100n^5(2n)^{n^2}} \deg(X)^{-(r+1)(n-r)-(sn+1)(2r)^{sn}(r+3)(n-r)} \delta.$$

Say  $\lambda \in \mathbf{N}$  such that  $\lambda\varphi \in \text{Mat}_{r,n}(\mathbf{Z})$ . We apply Proposition 8.2(ii) to get  $\deg(\lambda\pi\varphi|_X) \geq |\det \lambda\pi\varphi B_L| = \lambda^r |\det \pi\varphi B_L|$ . Since  $\deg(\lambda\pi\varphi|_X) = \lambda^r \deg(\pi\varphi|_X)$  we have  $\deg(\pi\varphi|_X) \geq |\det \pi\varphi B_L|$ .

The proposition now follow from (8.8).  $\square$

## 9. GENERICALLY BOUNDED HEIGHT

As usual, let  $r, s, n \in \mathbf{N}$ . For brevity we introduce a constant

$$\mu(r, s, n) = (r^2 + r + 1) \left( 1 + \frac{n-r}{2} (r+1 + (r+3)(sn+1)(2r)^{sn}) \right) + r.$$

It is convenient to allow  $r$  or  $s$  to be zero and set  $\mu(r, s, n) = 0$  in this case.

For example in the case of curves  $r = 1$  and if  $s = 1$ , then

$$\mu(1, 1, n) = 6(n^2 - 1)2^n + 3n + 1.$$

**Theorem 13.** *Let  $n \geq 1$  and let  $X \subset \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  and say  $s \geq 0$  is an integer. Then one of the following two cases holds.*

(i) *There exists an algebraic subgroup  $H \subset \mathbf{G}_m^n$  such that*

$$\dim_p X \cap pH \geq \max\{1, s + \dim H - n + 1\}$$

*for all  $p \in X(\overline{\mathbf{Q}})$ .*

(ii) *We set  $C = 2^{200n^7(2n)^{n^2}}$  and  $\mu = \mu(\dim X, s, n)$ . There is a non-empty Zariski open subset  $U \subset X$  such that if  $p \in U(\overline{\mathbf{Q}}) \cap (\mathbf{G}_m^n)^{[s]}$ , then*

$$h(p) \leq C \deg(X)^\mu (1 + h(X)).$$

Say  $p \in (\mathbf{G}_m^n)^{[s]}$ . Our approach is to find a surjective homomorphism  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^s$  from a finite set that maps  $p$  to a point of height not too large controlled with  $h(p)$ . Recall that we identify elements of  $\text{Mat}_{s,n}(\mathbf{Z})$  with homomorphisms  $\mathbf{G}_m^n \rightarrow \mathbf{G}_m^s$ .

**Lemma 9.1.** *Suppose  $1 \leq s \leq n$  and let  $Q \geq 2s!$  be a real number. If  $p \in (\mathbf{G}_m^n)^{[s]}$  there exists  $\psi \in \text{Mat}_{s,n}(\mathbf{Z})$  with*

$$|\pi\psi|_\infty \geq Q \quad \text{for all } \pi \in \Pi_{rs}, \quad |\psi|_\infty < Q + 1, \quad h(\psi(p)) \leq \frac{1}{2} \log(n+1) + (sn)h(p),$$

*and such that  $\psi/Q$  is  $1/(2s!)$ -regular.*

*Proof.* There is  $\psi_0 \in \text{Mat}_{s,n}(\mathbf{Z})$  of rank  $s$  whose kernel contains  $p$ .

An  $s \times s$  minor of  $\psi_0$  whose discriminant has maximal absolute value among all  $s \times s$  minors is invertible. We let  $\alpha^{-1}$  denote such a minor, then  $\alpha \in \text{Mat}_s(\mathbf{Q})$ . By our choice  $|\alpha\psi_0|_\infty = 1$  and some  $s \times s$  minor of  $\alpha\psi_0$  is the identity matrix.

There is  $\psi \in \text{Mat}_{s,n}(\mathbf{Z})$  with  $|Q\alpha\psi_0 - \psi|_\infty < 1$ . We may arrange that the sup-norm of each row of  $\psi$  is at least  $Q$ . So  $\psi$  satisfies the first property.

The second property follows from  $|\psi|_\infty \leq |Q\alpha\psi_0 - \psi|_\infty + |Q\alpha\psi_0|_\infty < Q + 1$ .

To prove the third claim we set  $\delta = \alpha\psi_0 - \psi/Q$  and fix  $\lambda \in \mathbf{N}$  with  $\lambda\alpha, \lambda\delta \in \text{Mat}_{s,n}(\mathbf{Z})$ . If  $u_1, \dots, u_s \in \mathbf{Z}^n$  are the rows of  $-\lambda\delta$ , then basic height properties imply

$$\begin{aligned} h_s((-\lambda\delta)(p)) &= h_s(p^{u_1}, \dots, p^{u_s}) \leq h_s(p^{u_1}) + \dots + h_s(p^{u_s}) \\ &\leq n(|u_1|_\infty + \dots + |u_s|_\infty)h_s(p) \leq sn|-\lambda\delta|_\infty h_s(p) = sn\lambda|\delta|_\infty h_s(p). \end{aligned}$$

By construction we have  $|\delta|_\infty \leq 1/Q$  and hence

$$(9.1) \quad h_s((-\lambda\delta)(p)) \leq sn \frac{\lambda}{Q} h_s(p).$$

But  $\lambda\psi = Q\lambda\alpha\psi_0 - Q\lambda\delta$  and  $\psi_0(p) = 1$ , so  $\lambda h_s(\psi(p)) = h_s((\lambda\psi)(p)) = Q h_s((-\lambda\delta)(p))$  by basic height properties. Using (9.1) we obtain  $h_s(\psi(p)) \leq sn h_s(p)$ . The third property follows from

$$h(\psi(p)) \leq \frac{1}{2} \log(n+1) + h_s(\psi(p)) \leq \frac{1}{2} \log(n+1) + sn h_s(p) \leq \frac{1}{2} \log(n+1) + sn h(p).$$

We turn to the final property. Let  $\phi \in \text{Mat}_{s,n}(\mathbf{R})$  satisfy  $|\phi|_\infty \leq \epsilon$  with  $\epsilon \leq 1/2$  and  $\text{rank}(\psi/Q + \phi) < s$ . The matrix  $\psi/Q + \phi = \alpha\psi_0 - (\delta - \phi)$  has an  $s \times s$ -minor which equals  $1 - (\delta' - \phi')$  where  $1$  is the unit matrix and  $\delta'$  and  $\phi'$  are  $s \times s$ -minors of  $\delta$  and  $\phi$ , respectively. We have  $|\delta' - \phi'|_\infty \leq |\delta|_\infty + |\phi|_\infty \leq 1/Q + \epsilon \leq 1$ . Using the Leibniz expansion for the determinant we conclude  $|\det(1 - (\delta' - \phi')) - 1|_\infty \leq s!|\delta' - \phi'|_\infty \leq$

$s!(1/Q + \epsilon) \leq 1/2 + s!\epsilon$  because  $Q \geq 2s!$ . If  $\epsilon < 1/(2s!)$ , then  $\det(1 - (\delta' - \phi')) \neq 0$  and in this case  $\psi/Q + \phi$  has full rank, contradicting our assumption on  $\phi$ . Thus we must have  $\epsilon \geq 1/(2s!)$ . It follows that  $\psi/Q$  is  $1/(2s!)$ -regular as desired.  $\square$

**Proposition 9.1.** *Let  $X \subsetneq \mathbf{G}_m^n$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$  of dimension  $r \geq 1$ . Let  $s$  be an integer with  $r \leq s \leq n$ , for brevity we write  $\mu = \mu(r, s, n)$ . Then we are either in alternative (i) of Proposition 8.1 or the following statement holds. There is a finite set of  $\{V_1, \dots, V_N\}$  of irreducible closed subvarieties of  $X$  with*

$$\dim V_i = r - 1,$$

$$\deg(V_1) + \dots + \deg(V_N) \leq 2^{200n^7(2n)^{n^2}} \deg(X)^{(r+1)\left(1+r\frac{\mu-r}{r^2+r+1}\right)},$$

$$h(V_i) \leq 2^{300n^7(2n)^{n^2}} \deg(X)^{2r+1+(2r^2+2r+1)\frac{\mu-r}{r^2+r+1}} (1 + h(X)),$$

and such that if  $p \in (X \setminus (V_1 \cup \dots \cup V_N))(\overline{\mathbf{Q}}) \cap (\mathbf{G}_m^n)^{[s]}$  then

$$h(p) \leq 2^{200n^7(2n)^{n^2}} \deg(X)^\mu (1 + h(X)).$$

*Proof.* We begin by choosing the parameter which appears in Lemma 9.1 as

$$Q = 2^{100n^5(2n)^{n^2}} \deg(X)^\chi$$

with

$$\chi = 1 + \frac{n-r}{2} (r+1 + (r+3)(sn+1)(2r)^{sn}) = \frac{\mu-r}{r^2+r+1} \geq 1.$$

Elementary estimates show  $Q \geq 2n! \geq 2s!$ .

Let  $p$  be as in the hypothesis and let  $\psi$  be given by Lemma 9.1 applied to  $p$ . We have  $|\psi|_\infty < Q + 1 \leq 2Q$ .

We apply Proposition 8.1 with  $\epsilon = 1/(2s!)$  to  $\psi/Q$ . Hence there is  $\pi \in \Pi_{rs}$  with

$$\deg(\varphi/Q|_X) \geq 2^{-50n^5(2n)^{n^2}} (4s!)^{-sn(2r)^{sn}} \deg(X)^{-\chi+1}$$

for  $\varphi = \pi\psi$ . We bound  $(4s!)^{sn(2r)^{sn}} \leq (4n!)^{n^2(2n)^{n^2}} \leq 2^{(n^2+2)n^2(2n)^{n^2}} \leq 2^{3n^4(2n)^{n^2}}$  using  $2n! \leq 2^{n^2}$ . So

$$(9.2) \quad \deg(\varphi/Q|_X) \geq 2^{-53n^5(2n)^{n^2}} \deg(X)^{-\chi+1}.$$

We set  $Z = \overline{X}^\varphi$  as in Section 7. This is an irreducible closed subvariety of  $\mathbf{P}^n \times \mathbf{P}^r$  defined over  $\overline{\mathbf{Q}}$  with dimension  $r$ . Further down we will apply Proposition 6.2 to  $Z$ . But first we bound the various quantities associated to  $Z$ . Indeed, Lemma 7.2 and  $Q \leq |\varphi|_\infty \leq 2Q$  give

$$\deg(\varphi|_X) \geq \kappa \geq \frac{Q}{(4n)^r \deg(X)} \frac{\deg(\varphi|_X)}{(2Q)^r} = \frac{Q}{(8n)^r \deg(X)} \deg(\varphi/Q|_X)$$

where  $\kappa = \kappa(Z)$ . Using the crude bound  $(8n)^r \leq (8n)^n \leq 2^{4n^2} \leq 2^{2n(2n)^{n^2}}$  together with (9.2) we obtain  $\kappa \geq 2^{-55n^5(2n)^{n^2}} Q \deg(X)^{-\chi}$ . On inserting our choice of  $Q$  we get

$$(9.3) \quad \kappa \geq 2^{45n^5(2n)^{n^2}}.$$

Lemma 7.1 enables us to bound

$$(9.4) \quad \Delta_0 = \Delta_0(Z) \leq (4n)^r (2Q)^r \deg(X) \leq (8n)^n Q^r \deg(X) \leq 2^{4n^2} Q^r \deg(X).$$

Degree and height of  $Z$  are bounded by Lemmas 7.1 and 7.3, we have

$$\deg(Z) \leq (8n)^n Q^r \deg(X) \leq 2^{4n^2} Q^r \deg(X)$$

and

$$h(Z) \leq (8n)^{n+2} (Q^{r+1} h(X) + Q^r \deg(X)) \leq 2^{12n^2} (Q^{r+1} h(X) + Q^r \deg(X)),$$

so

$$h(Z) + \deg(Z) \leq 2^{12n^2} Q^{r+1} (h(X) + 2\deg(X)/Q) \leq 2^{12n^2} Q^{r+1} (1 + h(X))$$

since  $Q \geq 2\deg(X)$ .

It is not difficult to verify  $\kappa \geq 17n^2 \geq 17rn$ . So the hypothesis on  $\kappa$  in Proposition 6.2 applied to  $Z$  is satisfied. We let  $V_1, \dots, V_N \subset X$  denote the obstruction varieties given by this proposition that actually meet  $\mathbf{G}_m^n$ .

Say  $k_0 = \max\{17 \cdot 3^r r! n \Delta_0(Z)^{r-1}, \deg(Z)\}$  is as in Proposition 6.2. We recall (9.4) and the bound for  $\deg(Z)$  to obtain

$$\begin{aligned} (9.5) \quad k_0 &\leq \max\{2^{5+2n} n! n 2^{4n^3} Q^{r(r-1)} \deg(X)^{r-1}, (8n)^n Q^r \deg(X)\} \\ &\leq 2^{13n^3} \max\{Q^{r(r-1)} \deg(X)^{r-1}, Q^r \deg(X)\} \\ &\leq 2^{13n^3} Q^{r^2} \deg(X)^r. \end{aligned}$$

Now  $\sum_{i=1}^N \deg(V_i) \leq k_0 \deg(Z)$ , so

$$\sum_{i=1}^N \deg(V_i) \leq 2^{4n^2} Q^r \deg(X) k_0 \leq 2^{17n^3} Q^{r^2+r} \deg(X)^{r+1}.$$

Our choice of  $Q$  and  $r^2 + r \leq (n-1)^2 + n - 1 \leq n^2$  implies

$$\sum_{i=1}^N \deg(V_i) \leq 2^{17n^3+100n^7(2n)^{n^2}} \deg(X)^{(r+1)(1+r\chi)}.$$

This bound leads quickly to the desired estimate for sum over the degrees of the  $V_i$ .

We let  $V$  denote one of the obstruction varieties  $V_i$ . We must bound  $h(V)$  too. Proposition 6.2 and the bounds for  $h(Z) + \deg(Z)$  and  $\deg(Z)$  give

$$\begin{aligned} h(V) &\leq 2^9 n^4 \max\left\{1, \frac{\kappa^{r+1}}{\Delta_0}\right\} (h(Z) + \deg(Z)) \deg(Z) k_0 \\ &\leq 2^{30n^2} Q^{2r+1} \Delta_0^r \deg(X) (1 + h(X)) k_0 \end{aligned}$$

in the last equality we used  $\kappa \leq \Delta_0$  which follows from Lemmas 7.2 and 7.1. Furthermore,  $\Delta_0$  is bounded by (9.4); this gives

$$h(V) \leq 2^{40n^3} Q^{r^2+2r+1} \deg(X)^{r+1} (1 + h(X)) k_0.$$

With (9.5) we get

$$h(V) \leq 2^{60n^3} Q^{r^2+2r+1} \deg(X)^{2r+1} (1 + h(X)).$$

Since  $r \leq n-1$  we have  $2r^2 + 2r + 1 \leq 2n^2$ , so with our choice of  $Q$  we find

$$h(V) \leq 2^{300n^7(2n)^{n^2}} \deg(X)^{2r+1+(2r^2+2r+1)\chi} (1 + h(X)),$$

as desired.

To complete the proof it remains to bound the height of  $p$ . Clearly, we may assume that  $h(p) \geq 1$ . No projective coordinate of  $p$  vanishes. If  $q = \varphi(p)$  then  $(p, q)$  is isolated in  $\pi_1|_Z$ ; indeed, locally at  $(p, q)$  the variety  $Z$  is the graph of  $\varphi : \mathbf{G}_m^n \rightarrow \mathbf{G}_m^r$ . Hence we are in case (iii) of Proposition 6.2. Thus, on using  $r \leq n$  and  $\kappa \geq 1$  we have

$$\begin{aligned} h(p) &\leq \frac{2^5 n^2}{\kappa} h(q) + 2^{15} n^5 \max \left\{ 1, \frac{\kappa^{r+1}}{\Delta_0} \right\} (h(Z) + \deg(Z)) \\ &\leq \frac{2^{7n}}{\kappa} h(q) + 2^{20n} \Delta_0^r (h(Z) + \deg(Z)) \end{aligned}$$

here we used  $\kappa \leq \Delta_0$  again. Upper bounds for  $\Delta_0$  and  $h(Z) + \deg(Z)$  were obtained above; we conclude

$$\begin{aligned} h(p) &\leq \frac{2^{7n}}{\kappa} h(q) + 2^{20n+4n^2r+12n^2} Q^{r^2+r+1} \deg(X)^r (1 + h(X)) \\ &\leq \frac{2^{7n}}{\kappa} h(q) + 2^{40n^3} Q^{r^2+r+1} \deg(X)^r (1 + h(X)) \end{aligned}$$

Our choice of  $Q$  and  $r^2 + r + 1 \leq (n-1)^2 + (n-1) + 1 \leq n^2$  implies

$$h(p) \leq \frac{2^{7n}}{\kappa} h(q) + 2^{150n^7(2n)^{n^2}} \deg(X)^{(r^2+r+1)\chi+r} (1 + h(X)).$$

We note that  $(r^2 + r + 1)\chi + r = \mu$ , so

$$h(p) \leq \frac{2^{7n}}{\kappa} h(q) + 2^{150n^7(2n)^{n^2}} \deg(X)^\mu (1 + h(X)).$$

Lemma 9.1 tells us that  $h(q) \leq h(\psi(p)) \leq \frac{1}{2} \log(n+1) + n^2 h(p) \leq n + n^2 h(p) \leq 2n^2 h(p) \leq 2^{3n} h(p)$  since  $h(p) \geq 1$ . We obtain

$$\left( 1 - \frac{2^{10n}}{\kappa} \right) h(p) \leq 2^{150n^7(2n)^{n^2}} \deg(X)^\mu (1 + h(X)).$$

Certainly, (9.3) implies  $\kappa \geq 2^{11n}$ . So  $1 - 2^{10n} \kappa^{-1} \geq 1/2$  and the proposition follows.  $\square$

*Proof of Theorem 13.* We may suppose  $s \leq n$  since  $(\mathbf{G}_m^n)^{[s]} = \emptyset$  otherwise. If  $r = 0$ , then part (ii) holds with  $U = X$  because  $\mu(0, s, n) = 0$  and since the height of a point is its height considered as a variety. So we assume  $r \geq 1$ . If  $s < r$ , then we are in case (i) with  $H = \mathbf{G}_m^n$ . So we may suppose  $r \leq s$ . If  $s = n$ , then  $(\mathbf{G}_m^n)^{[s]}$  is the set of torsion points of  $\mathbf{G}_m^n$ . The height of a torsion point is  $\frac{1}{2} \log(n+1) \leq n$  and so the height bound in part (ii) holds with  $U = X$ . So let us assume  $s \leq n-1$ . Then  $1 \leq r \leq n-1$  and the theorem follows from the previous proposition.  $\square$

## 10. BOUNDED HEIGHT

In this section we prove Theorem 11.

The first few reduction steps are similar as in the proof of Theorem 13.

We may suppose  $s \leq n$  since  $(\mathbf{G}_m^n)^{[s]} = \emptyset$  otherwise. If  $r = 0$ , then part (ii) holds with  $Z = \emptyset$  because  $\mu(0, s, n) = 0$  and since the height of a point is its height considered as a variety. So we assume  $r \geq 1$ . If  $s < r$ , then  $X^{\text{oa}, [s]} = \emptyset$  and the theorem follows with  $Z = X$ . So we may suppose  $r \leq s$ . If  $s = n$ , then  $(\mathbf{G}_m^n)^{[s]}$  is the set of torsion points of

$\mathbf{G}_m^n$ . The height of a torsion point is  $\frac{1}{2} \log(n+1) \leq n$  and so the height bound in part (iii) holds with  $Z = \emptyset$ . So let us assume  $s \leq n-1$ .

We have reduced to the case  $1 \leq r \leq n-1$ .

For brevity, we set  $\mu = \mu(r, s, n)$ . Elementary estimates lead to

$$(10.1) \quad \mu \leq n^2 \left( 1 + \frac{n-1}{2} \left( n + n^2(n+2)(2n)^{n^2-n} \right) \right) + n - 1 \leq n^6(2n)^{n^2}.$$

We remark  $\mu \geq 2$  and  $r^2 + r + 1 \geq 2r$ . We use  $1 \leq r \leq n-1$  and (10.1) to bound the exponents

$$(10.2) \quad (r+1) \left( 1 + r \frac{\mu - r}{r^2 + r + 1} \right) \leq n \left( 1 + \frac{\mu}{2} \right) \leq n\mu \leq n^7(2n)^{n^2},$$

$$(10.3) \quad 2r + 1 + (2r^2 + 2r + 1) \frac{\mu - r}{r^2 + r + 1} \leq 2n + 5r^2 \frac{\mu}{2r} \leq 5n\mu \leq 5n^7(2n)^{n^2}.$$

that appear in the degree and height bound of Proposition 9.1, respectively.

The remainder of the proof is a somewhat technical descent argument. For  $0 \leq i \leq \dim X$  we will inductively construct irreducible closed subvarieties  $V_{j_1, \dots, j_i}^{(i)}$  of  $X$  which satisfy certain properties to be described below; here  $(j_1, \dots, j_i)$  runs over a finite index set

$$(10.4) \quad \{(j_1, \dots, j_i); 1 \leq j_1 \leq N^{(0)}, 1 \leq j_2 \leq N_{j_1}^{(1)}, \dots, 1 \leq j_i \leq N_{j_1, \dots, j_{i-1}}^{(i-1)}\}.$$

The properties for the subvarieties are as follows

$$(10.5) \quad \dim V_{j_1, \dots, j_i}^{(i)} = \dim X - i,$$

$$(10.6)$$

$$\deg(V_{j_1, \dots, j_{i-1}, 1}^{(i)} + \dots + \deg(V_{j_1, \dots, j_{i-1}, N_{j_1, \dots, j_{i-1}}^{(i-1)}}^{(i)}) \leq 2^{(300n^7(2n)^{n^2})^i} \deg(X)^{(n^7(2n)^{n^2})^i},$$

$$(10.7) \quad h(V_{j_1, \dots, j_i}^{(i)}) \leq 2^{(600n^7(2n)^{n^2})^i} \deg(X)^{(6n^7(2n)^{n^2})^i} (1 + h(X)),$$

$$(10.8) \quad V_{j_1, \dots, j_i}^{(i)} \subset V_{j_1, \dots, j_{i-1}}^{(i-1)} \quad \text{if } i \geq 1.$$

For  $i = 0$  we take  $V^{(0)} = X$ . Clearly, (10.5), (10.6), and (10.7) are satisfied. So we suppose  $i \geq 1$  and that  $V_{j_1, \dots, j_{i-1}}^{(i-1)}$  has been constructed.

If  $V_{j_1, \dots, j_{i-1}}^{(i-1)}$  is as in alternative (i) of Proposition 8.1 then we set  $N_{j_1, \dots, j_{i-1}}^{(i-1)} = 0$  and the construction stops. Otherwise, we take  $N_{j_1, \dots, j_{i-1}}^{(i-1)} = N$  as in Proposition 9.1 and  $V_{j_1, \dots, j_{i-1}, 1}^{(i)}, \dots, V_{j_1, \dots, j_{i-1}, N_{j_1, \dots, j_{i-1}}^{(i-1)}}^{(i)}$  to be the subvarieties  $V_1, \dots, V_N$  constructed there.

Properties (10.5) and (10.8) follow immediately from this proposition. It remains to prove degree and height bounds; this is done by an elementary calculation.

We begin by verifying (10.6). To do this, we keep (10.2) in mind. By Proposition 9.1, the left-hand side of (10.6) is at most

$$2^{200n^7(2n)^{n^2}} \deg(V_{j_1, \dots, j_{i-1}}^{(i-1)}) n^7(2n)^{n^2} \leq 2^{200n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^{i-1} n^7(2n)^{n^2}} \deg(X)^{(n^7(2n)^{n^2})^i}$$

on using the induction hypothesis. Elementary estimates and  $i \geq 1$  give

$$200n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^{i-1} n^7(2n)^{n^2} \leq (300n^7(2n)^{n^2})^i$$

so (10.6) follows.

Now we shall bound the height. We recall (10.3), so

$$h(V_{j_1, \dots, j_i}^{(i)}) \leq 2^{300n^7(2n)^{n^2}} \deg(V_{j_1, \dots, j_{i-1}}^{(i-1)}) 5n^7(2n)^{n^2} (1 + h(V_{j_1, \dots, j_{i-1}}^{(i-1)}))$$

by Proposition 9.1. We insert both degree and height bounds from the induction hypothesis to find that  $h(V_{j_1, \dots, j_i}^{(i)})$  is at most

$$2^{300n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^{i-1} 5n^7(2n)^{n^2} + (600n^7(2n)^{n^2})^{i-1}} \deg(X)^{(n^7(2n)^{n^2})^{i-1} 5n^7(2n)^{n^2} + (6n^7(2n)^{n^2})^{i-1}} (1 + h(X)).$$

Basic estimates and  $i \geq 1$  lead to

$$300n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^{i-1} 5n^7(2n)^{n^2} + (600n^7(2n)^{n^2})^{i-1} \leq (600n^7(2n)^{n^2})^i$$

as well as

$$(n^7(2n)^{n^2})^{i-1} 5n^7(2n)^{n^2} + (6n^7(2n)^{n^2})^{i-1} \leq (6n^7(2n)^{n^2})^i.$$

Hence claim (10.7) holds true.

We define

$$Z = \bigcup_{i=0}^r \bigcup_{\substack{V_{j_1, \dots, j_i}^{(i)} \\ \text{as in alt. (i) of Proposition 8.1}}} V_{j_1, \dots, j_i}^{(i)}.$$

It is a Zariski closed subset of  $X$ . This will be the Zariski closed set referred to in the assertion.

We have  $X = Z$  if and only if  $V^{(0)}$  appears in the union above. The degree and height bound for the irreducible components of  $Z$  follows from (10.6) and (10.7).

We now verify that the bound for the number of irreducible components of  $Z$  is correct. If  $V^{(0)} = X$  is an irreducible component of  $Z$ , then our bound certainly holds. Otherwise, an irreducible component is of the form  $V_{j_1, \dots, j_i}^{(i)}$  for some  $1 \leq i \leq r$ . For fixed  $i$  and  $j_1, \dots, j_{i-1}$ , the number of possible  $j_i$  is at most  $N_{j_1, \dots, j_{i-1}}^{(i-1)}$ . This quantity is bounded from above by (10.6). Recalling (10.4) we find that for fixed  $i$  there are at most

$$2^{300n^7(2n)^{n^2} + \dots + (300n^7(2n)^{n^2})^i} \deg(X)^{n^7(2n)^{n^2} + \dots + (n^7(2n)^{n^2})^i} \leq 2^{r(300n^7(2n)^{n^2})^r} \deg(X)^{r(n^7(2n)^{n^2})^r}$$

possible  $V_{j_1, \dots, j_i}^{(i)}$ . To get a bound for all possible irreducible components we must sum over  $1 \leq i \leq r$ . This gives us the bound

$$r 2^{r(300n^7(2n)^{n^2})^r} \deg(X)^{r(n^7(2n)^{n^2})^r} \leq 2^{r+r(300n^7(2n)^{n^2})^r} \deg(X)^{r(n^7(2n)^{n^2})^r}.$$

Part (ii) of the theorem follows since  $r+r(300n^7(2n)^{n^2})^r \leq (600n^7(2n)^{n^2})^r$  and  $r(n^7(2n)^{n^2})^r \leq (2n^7(2n)^{n^2})^r$ .

We claim that  $X^{\text{oa}, [s]} \subset X \setminus Z$ . This will imply part (i) of the theorem. So suppose  $p \in Z(\overline{\mathbf{Q}})$ . By definition of  $Z$  there are  $i, j_1, \dots, j_i$  and an algebraic subgroup  $H \subset \mathbf{G}_m^n$  such that  $\dim_p V_{j_1, \dots, j_i}^{(i)} \cap pH \geq \max\{1, s + \dim H - n + 1\}$ . Certainly, this local dimension is at most  $\dim_p X \cap pH$ . So  $p \notin X^{\text{oa}, [s]}$  and our claim is established.

Suppose  $p \in (X \setminus Z)(\overline{\mathbf{Q}})$  with  $p \in (\mathbf{G}_m^n)^{[s]}$ . It remains to bound  $h(p)$  as in (iii) of the theorem.

By construction  $X = V^{(0)}$ . We may choose  $0 \leq i \leq r$  maximal, such that there are indices  $j_1, \dots, j_i$  with  $p \in V_{j_1, \dots, j_i}^{(i)}(\overline{\mathbf{Q}})$ . We split up into two cases.



First, let us assume  $i = r$ . This case is easy, since  $V_{j_1, \dots, j_i}^{(i)} = \{p\}$  holds by (10.5). Now  $h(p)$  is the height of the variety  $\{p\}$ , so the desired height bound follows from (10.7).

Now, we suppose  $0 \leq i \leq r - 1$ . We remark that  $p \notin Z(\overline{\mathbf{Q}})$  implies that  $V_{j_1, \dots, j_i}^{(i)}$  is not as in alternative (i) of Proposition 8.1. In particular, Proposition 9.1 gives a height bound for  $p$  providing it does not lie on

$$V_{j_1, \dots, j_i, 1}^{(i+1)} \cup \dots \cup V_{j_1, \dots, j_i, N_{j_1, \dots, j_i}^{(i)}}^{(i+1)}.$$

But  $p$  cannot lie in this union because of the maximality of  $i$ . Recalling (10.1) gives us

$$h(p) \leq 2^{200n^7(2n)^{n^2}} \deg(V_{j_1, \dots, j_i}^{(i)})^{n^6(2n)^{n^2}} (1 + h(V_{j_1, \dots, j_i}^{(i)})).$$

Our degree bound (10.6) and height bound (10.7) give

$$2^{200n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^i n^6(2n)^{n^2} + (600n^7(2n)^{n^2})^i} \deg(X)^{(n^7(2n)^{n^2})^i n^6(2n)^{n^2} + (6n^7(2n)^{n^2})^i} (1 + h(X))$$

as a bound for  $h(p)$ . The exponent of 2 is

$$200n^7(2n)^{n^2} + (300n^7(2n)^{n^2})^i n^6(2n)^{n^2} + (600n^7(2n)^{n^2})^i \leq (200 + 300^{r-1} + 600^{r-1})(n^7(2n)^{n^2})^r.$$

because  $i \leq r - 1$ . Hence it is at most  $(600n^7(2n)^{n^2})^r$ . The exponent of  $\deg(X)$  is

$$(n^7(2n)^{n^2})^i n^6(2n)^{n^2} + (6n^7(2n)^{n^2})^i \leq (1 + 6^{r-1})(n^7(2n)^{n^2})^r \leq (6n^7(2n)^{n^2})^r.$$

Therefore,  $h(p) \leq 2^{(600n^7(2n)^{n^2})^r} \deg(X)^{(6n^7(2n)^{n^2})^r} (1 + h(X))$  and part (iii) of the theorem holds.  $\square$

## APPENDIX A. THE CASE OF ABELIAN VARIETIES

An abelian variety  $A$  defined over  $\overline{\mathbf{Q}}$  together with an ample symmetric line bundle determines a height function called the Néron-Tate or canonical height. It is a quadratic form and vanishes precisely on the torsion points of  $A$ . We refer to Chapter 9 of Bombieri and Gubler's book [4] for the necessary background.

The history of height upper bounds on subvarieties of abelian varieties runs parallel to the history of height bounds on the algebraic torus. But we will only give a brief account of what is known in the projective case.

An initial result was obtained by Viada [45] who proved the following analog of Theorem 1.

**Theorem 14** (Viada [45]). *Let  $A = E^g$  where  $E$  is an elliptic curve defined over  $\overline{\mathbf{Q}}$  and suppose that we have fixed a symmetric line bundle, and thus a Néron-Tate height, on  $A(\overline{\mathbf{Q}})$ . Let  $C \subset A$  be an irreducible algebraic curve that is not contained in the translate of a proper algebraic subgroup of  $A$ . Then the height of points on  $C$  that are contained in a proper algebraic subgroup is bounded from above uniformly.*

Rémond [36] gave a systematic approach for passing from height upper bounds in the spirit of Theorem 14 to finiteness result using Lehmer-type and relative Lehmer-type height lower bounds. These inequalities remain conjectural for many abelian varieties. For example, no sufficiently strong Lehmer-type height inequality is known on a power of an elliptic curve without complex multiplication to tackle the abelian analog of Theorem 2 using Rémond's approach. Viada [45] did obtain a finiteness result akin to Theorem 2 when the elliptic curve in question has complex multiplication. In this setting the

sufficiently strong Lehmer-type height lower bounds are available thanks to work of David and Hindry [14].

The analog of Maurin's Theorem for curves inside a power of an elliptic curve was obtained already in 2003 by Rémond and Viada [39]. Again the elliptic curve was assumed to have complex multiplication. As Maurin's Theorem, Rémond and Viada's Theorem relies on Rémond's Generalized Vojta Inequality [35].

Advances made primarily by Galateau [18, 19] on Bogomolov-type height lower bounds have had a catalytic effect on the finiteness problems. His results hold for a wide class of abelian varieties, including abelian surfaces and arbitrary powers of elliptic curves. Viada [46] used them to prove the analog of Maurin's Theorem for curves defined over  $\overline{\mathbf{Q}}$  inside a power of an arbitrary elliptic curve defined over  $\overline{\mathbf{Q}}$ .

Partial results for subvarieties of arbitrary dimension in an abelian variety are known as well. Here the definition of  $X^{\text{oa}}$  for a subvariety  $X \subset A$  is verbatim to the toric case. The union  $A^{[s]}$  of all algebraic subgroups of  $A$  of codimension at least  $s$  also makes perfect sense. For example, the author proved [21] the full analog of Theorem 9. That is, the Néron-Tate height is uniformly bounded from above on  $X^{\text{oa}}(\overline{\mathbf{Q}}) \cap A^{[\dim X]}$ .

Rémond's Generalized Vojta Inequality is powerful enough to treat, along with varying algebraic subgroups, the division closure of a finite rank subgroup  $\Gamma \subset A(\overline{\mathbf{Q}})$ . Indeed, he considers points on  $X$  contained in

$$A^{[s]} + \Gamma = \{h + \gamma; h \in A^{[s]} \text{ and } \gamma \in \Gamma\}.$$

**Theorem 15** (Rémond [38], cf. [37]). *Let  $A$  be an abelian variety defined over  $\overline{\mathbf{Q}}$  and suppose we have fixed a Néron-Tate height on  $A$ . Let  $\Gamma \subset A(\overline{\mathbf{Q}})$  be the division closure of a finitely generated subgroup of  $A$ . Moreover, let  $X \subset A$  be an irreducible closed subvariety defined over  $\overline{\mathbf{Q}}$ . Then the height of points in  $X^{\text{oa}}(\overline{\mathbf{Q}}) \cap (A^{[1+\dim X]} + \Gamma)$  is bounded from above uniformly.*

Maurin's Theorem 10 is the toric version of this result.

## APPENDIX B. HEIGHT BOUNDS IN SHIMURA VARIETIES

The well-known analogy between semi-abelian varieties and Shimura varieties which is underlined by the conceptual similarity of the Conjectures of Manin-Mumford and André-Oort tempts us to formulate a Bounded Height Conjecture in the moduli-theoretic setting. The role of algebraic subgroups on the abelian or toric side is played by the special subvarieties on the side of Shimura varieties. Furthermore, the torsion points are replaced by special points. The sweeping conjecture of Pink [34] on mixed Shimura varieties covers the Conjectures of Manin-Mumford and André-Oort. By the comments in Bombieri, Masser and Zannier's appendix [10], Pink's Conjecture implies Zilber's Conjecture 1 [49] and Bombieri, Masser, and Zannier's Torsion Finiteness Conjecture [8].

However, a too literal generalization of the Bounded Height Conjecture to Shimura varieties is false. Bombieri, Masser, and Zannier [8] showed that the height of the  $j$ -invariant of an elliptic curve with complex multiplication can be arbitrary large. These  $j$ -invariants, which we call singular moduli, are algebraic integers. The Shimura variety in question is  $Y(1)$ , the modular curve whose complex points correspond to the  $j$ -invariants of elliptic curves defined over  $\mathbf{C}$ . As a variety  $Y(1)$  equals the affine line. Unboundedness

of height already follows from an earlier more general result of Colmez [13] who proved the following estimate.

The discriminant of a singular moduli is the discriminant of the endomorphism ring of the corresponding elliptic curve.

**Theorem 16** (Colmez [13]). *There exists an absolute constant  $c > 0$  with the following property. If  $j$  is a singular moduli whose discriminant  $\Delta$  is a fundamental discriminant, then*

$$h(j) \geq -c^{-1} + c \log |\Delta|.$$

Polynomial upper bounds in the discriminant are available using classical estimates in analytic number theory. Pila and the author proved the following inequality.

**Lemma B.1** (Lemma 4.3 [24]). *For any  $\epsilon > 0$  there exists a constant  $c > 0$  with the following property. If  $j$  is a singular moduli with discriminant  $\Delta$ , then*

$$h(j) \leq c|\Delta|^\epsilon.$$

If the Generalized Riemann Hypothesis (GRH) is true, then one may replace  $|\Delta|^\epsilon$  in this upper bound by  $\log |\Delta|$ . We refer to Lemmas 3 and 5 [23] for an even better estimate.

The special subvarieties of the product  $Y(1)^2$  are known. Points whose coordinates are both singular moduli are precisely the special points and  $Y(1)^2$  itself is the only two-dimensional special subvariety. Among the special curves we find the vertical and horizontal lines where the fixed coordinate is a singular moduli. The remain ones are  $Y_0(N) \subset Y(1)^2$  and given by the zero-sets of the classical  $N$ th modular transformation polynomial for  $N \in \mathbf{N}$ . A complex point on such a special curve corresponds to a pair of elliptic curves that are linked by an isogeny of degree  $N$  with cyclic kernel.

If  $C \subset Y(2)^2$  is not a special curve then the author proved [23] that there is a constant  $c > 0$  such that  $C \cap Y_0(p)$  contains a point of height at least  $c \log p$  for all primes  $p \geq c^{-1}$ . Hence not even  $C \cap \bigcup_{N \geq 1} Y_0(N)$  has bounded height. The situation already looks dire in a product of two modular curves.

As there can be no such thing as a Bounded Height Conjecture in the Shimura setting we must content ourselves with something less. In the particular case of curves in  $Y(1)^2$  the author formulated the following conjecture.

If  $S \subset Y(1)^2$  is an irreducible curve defined over  $\overline{\mathbf{Q}}$  then we let  $\deg_{\mathbf{Q}} S$  denote the degree of the union of all conjugates of  $S$  over  $\mathbf{Q}$ . We observe that any point  $p \in Y(1)^2$  is contained in a uniquely determined *minimal* special subvariety  $\mathcal{S}(p) \subset Y(2)^2$ . For example, if  $p = (j, *)$  is a special point, then  $\mathcal{S}(p) = \{p\}$  and  $\deg_{\mathbf{Q}} \mathcal{S}(p) = [\mathbf{Q}(p) : \mathbf{Q}] \geq [\mathbf{Q}(j) : \mathbf{Q}]$ . If  $\epsilon > 0$  then, by the Siegel-Brauer Theorem,  $[\mathbf{Q}(j) : \mathbf{Q}]$  grows at least of the order  $|\Delta|^{1/2-\epsilon}$  where  $\Delta$  is the discriminant of the singular moduli  $j$ .

**Conjecture** (Weakly Bounded Height Conjecture for  $Y(1)^2$  [23]). *Let  $C \subset Y(1)^2$  be an irreducible algebraic curve defined over  $\overline{\mathbf{Q}}$  that is not special. There exists a constant  $c > 0$  with the following property. Suppose  $p \in C$  and  $\dim \mathcal{S}(p) \leq 1$ , then*

$$(B.1) \quad h(p) \leq c \log(1 + \deg_{\mathbf{Q}} S).$$

By Corollary 1.2(ii) [23] the GRH implies this conjecture for a class of curves satisfying a geometric restriction in addition to being non-special. Part (i) of this corollary implies

that we can still obtain a logarithmic height bound (B.1) without assuming the GRH if we restrict to points satisfying  $\mathcal{S}(p) = Y_0(N)$  for some  $N \in \mathbf{N}$ . GRH is used solely to obtain an upper for the height of a singular moduli that is logarithmic in terms of its discriminant. But as we have seen in Lemma B.1, polynomials upper bounds with arbitrarily small exponent hold unconditionally.

The following, even weaker, bounded height conjecture could bail us out should the GRH default.

**Conjecture** (Super Weakly Bounded Height Conjecture for  $Y(1)^2$ ). *Let  $\epsilon > 0$  and let  $C \subset Y(1)^2$  be an irreducible algebraic curve defined over  $\overline{\mathbf{Q}}$  that is not special. There exists a constant  $c > 0$  with the following property. Suppose  $p \in C$  and  $\dim \mathcal{S}(p) \leq 1$ , then*

$$h(p) \leq c(\deg_{\mathbf{Q}} S)^\epsilon.$$

Already this conjecture has implications in direction of Pink's Conjecture. Let us suppose for the moment that it holds. Using arguments laid out in the author's joint work with Pila [24] one can show the following finiteness statement. Suppose  $C \subset Y(1)^n$  is an irreducible algebraic curve defined over  $\overline{\mathbf{Q}}$  that is not contained in a proper special subvariety of  $Y(1)^n$ . Then  $C$  contains only finitely many points that are inside a special subvariety of  $Y(1)^n$  of codimension at least 2.

Theorem 1 [24] implies this finiteness result unconditionally for curves satisfying an additional geometric hypothesis.

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